

This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Ed. T 149.17.334



Harvard College Library
THE GIFT OF
GINN AND COMPANY





3 2044 097 046 601

PROJECTIVE GEOMETRY

MODERN MATHEMATICAL TEXTS

EDITED BY

CHARLES S. SLICHTER

ELEMENTARY MATHEMATICAL ANALYSIS
By CHARLES S. SLICHTER
490 *pages*, 5 x 7½, *Illustrated* $2.50

MATHEMATIC FOR AGRICULTURAL STUDENTS
By HENRY C. WOLFF
311 *pages*, 5 x 7½, *Illustrated* , ... $1.50

CALCULUS
By HERMAN W. MARCH AND HENRY C. WOLFF
360 *pages*, 5 x 7½, *Illustrated* $2.00

PROJECTIVE GEOMETRY
By L. WAYLAND DOWLING
316 *pages*,5 x 7½, *Illustrated* $2.00

MODERN MATHEMATICAL TEXTS

EDITED BY CHARLES S. SLICHTER

PROJECTIVE GEOMETRY

BY

L. WAYLAND DOWLING, PH. D.
ASSOCIATE PROFESSOR OF MATHEMATICS
UNIVERSITY OF WISCONSIN

FIRST EDITION

McGRAW-HILL BOOK COMPANY, INC.
239 WEST 39TH STREET. NEW YORK

LONDON: HILL PUBLISHING CO., LTD.
6 & 8 BOUVERIE ST., E. C.

1917

Educ T 149.17.334

HARVARD COLLEGE LIBRARY
GIFT OF
GINN & CO.
JUL 24 1941

COPYRIGHT, 1917, BY THE
McGRAW-HILL BOOK CO., INC.

THE MAPLE PRESS YORK PA

PREFACE

The present volume embodies a course of lectures on Projective Geometry given by the author for a number of years at the University of Wisconsin. The synthetic point of view was chosen primarily to develop the power of visualization and of pure geometric analysis for young men and women preparing to teach geometry in our secondary schools. Such a course should naturally avoid a review of the subject matter of Elementary Geometry and, at the same time, should not be so far removed from familiar concepts as to lose connection with them. In the second place, the synthetic treatment of loci of the second order and of the second class opens up a new field to the student familiar with analytical processes and has certain advantages in arousing his enthusiasm for continued work in mathematics.

No especial preparation beyond Elementary Geometry and a slight knowledge of Trigonometry is required in order to read this book with perfect understanding. The reader who knows his Analytic Geometry will often find himself on familiar ground, but no knowledge beyond the use of coördinate axes is assumed.

The book is frankly patterned after Reye's *Geometrie der Lage*, with the feeling that the general method of treatment adopted by Professor Reye best serves the purposes outlined above. On the other hand, the author has not failed to consult and to profit by other texts on Projective Geometry that occupy important places in recent literature; notably, Veblen and Young, *Projective Geometry;* Enriques, *Geometria Proiettiva;* Severi, *Complementi di Geometria Proeittiva.*

No attempt has been made to set forth a necessary and sufficient set of postulates for Projective Geometry; not that the author fails to recognize the importance of research already completed in this field, but because of the conviction that the student is unfitted to appreciate work of this character until he has assimilated the main body of theorems and their applications based upon concepts familiar to him from the study of Elementary Geometry. This,

too, is in accord with the aims set forth above. The existence of ideal elements must be assumed (Art. 7); and the Dedekind postulate, or an equivalent, must be used in order to arrive at continuously projective forms. The treatment of the Dedekind postulate for this purpose (Art. 39) is confessedly meager, and many teachers may feel the need of expanding it, or indeed of restating it, as occasion seems to demand.

No attempt has been made to introduce new or strange terms, the only exception, so far as the author is aware, is the use of the word "confocal" to indicate those elements of a double polarity which are the supports of coinciding involutions of conjugate elements (Art. 159).

While this book has grown out of lectures given to students preparing to teach geometry, the subject matter is by no means of interest to this class of students alone. The engineer and the artisan must of necessity become familiar with the elementary processes of projection and section, and these processes are the same whether they lead to properties of geometrical figures or to methods in mechanical drawing.

The author takes this occasion to express his gratitude to Professor Thomas F. Holgate, now Acting President of Northwestern University, for inspiration and enthusiasm acquired under his instruction at Clark University; and also his indebtedness for many helpful suggestions during the preparation of the manuscript for this book.

Especial acknowledgment is due to Professor Henry S. White, of Vassar College, who read the manuscript and whose kindly comments and criticisms have materially improved the book in a number of points.

The author wishes also to express his thanks to Professor Charles S. Slichter, of the University of Wisconsin for his sympathetic interest during the preparation of the manuscript and for his aid in seeing the book through the press.

L. Wayland Dowling.

University of Wisconsin,
June, 1917.

CONTENTS

PAGE

PREFACE v

CHAPTER I

THE ELEMENTS AND THE PRIMITIVE FORMS

1. Central Projection 1
2. The Elements 1
3. Notation 2
4. Primitive Forms 2
5. Projection and Section 3
6. Projection and Section Applied to the Primitive Forms . . 4
7. Ideal Elements 5

CHAPTER II

THE PRINCIPLE OF DUALITY—SIMPLE AND COMPLETE FIGURES

8. Duality in the Plane 8
9. Duality in Space 9
10. The Principle of Duality 10
11. Simple Figures in a Plane 10
12. Complete Figures in a Plane 12
13. Simple and Complete Figures in Space 13

CHAPTER III

CORRELATION OF GEOMETRIC FIGURES; PERSPECTIVE POSITION OF GEOMETRIC FIGURES

14. Correlation of Geometric Figures 16
15. Perspective Position of Geometric Figures 16
16. Notation 19
17. Desargues Theorem—Theorem I 20
18. Converse of Theorem I 22
19. Definitions 22

PAGE
20. Complete Quadrangles in Perspective Position—Theorem II 23

CHAPTER IV

HARMONIC RANGES AND HARMONIC PENCILS

21. Harmonic Ranges 25
22. Effect of Order. 26
23. Construction of Harmonic Ranges—Theorem III. 27
24. Harmonic Pencils 28
25. Sections of Harmonic Pencils—Theorem IV 28
26. Harmonic Conjugate of an Ideal Point 30
27. Normal Conjugate Rays in a Harmonic Pencil. 30
28. Cross-ratio of a Harmonic Range. 31
29. Harmonic Mean Between Two Numbers 32
30. Geometric Mean Between Two Numbers 33
31. The Circle of Apollonius 34
32. Orthogonal Circles 34

CHAPTER V

PROJECTIVELY RELATED PRIMITIVE FORMS OF THE FIRST KIND

33. Primitive Forms of the First Kind 37
34. Chains of Perspectivity 37
35. Definition of Projective Relationship 38
36. Harmonic Scales 38
37. Two Pairs of Points Each Harmonically Separated by a Third Pair—Theorem V. 39
38. Converse of Theorem V. 40
39. Consequences of Theorem V and Its Converse—Theorem VI. 41
40. Superposition—Self-corresponding Elements 44
41. Von Staudt's Fundamental Theorem—Theorem VII . . . 46
42. Consequences of Theorem VII 47
43. Determination of Projective Relationship—Theorem VIII. 50

CHAPTER VI

ELEMENTARY FORMS

44. Definition of Elementary Forms 55

PAGE
45. Fundamental Properties of the Curve and the Envelope. . 57
46. Construction of Curves and Envelopes 62
47. Relations Existing among the Elementary Forms. 64
48. Totality of Elementary Forms 66
49. Classification of Curves of Second Order 66
50. The Conic Sections. 66

CHAPTER VII

THE PASCAL THEOREM AND THE BRIANCHON THEOREM

51. Six Elements of a Conic or of an Envelope—Theorems IX. 68
52. Converse Theorems. 69
53. Application of Theorems IX. 71
54. Degenerate Cases of Theorems IX 72
55. The Pentagon Theorem and Its Dual. 73
56. Application of the Pentagon Theorem and Its Dual. . . . 74
57. The Quadrangle Theorem and Its Dual 76
58. Application of the Quadrangle Theorem and Its Dual. . . 76
59. The Principle of Continuity 77
60. Second Proof of the Quadrangle Theorem and Its Dual . . 77
61. Generation of Particular Conics and Envelopes 78
62. Cones and Sheaves of Planes of Second Class 79
63. Cylinders 79

CHAPTER VIII

POLES AND POLAR LINES WITH RESPECT TO A CURVE OF SECOND ORDER

64. Poles and Polar Lines—Theorems X. 81
65. Special Positions of Pole and Polar Line. 83
66. Chords of Contact 83
67. Construction of Poles and Polar Lines 84
68. Conjugate Points and Conjugate Lines with Respect to a Conic—Theorems XI. 86
69. Consequences of Theorems XI. 87
70. Polar Figures with Respect to a Fixed Conic. 89
71. Self-polar Figures. 91
72. Pole-rays and Polar Planes with Respect to a Cone. . . . 91

CHAPTER IX

PAGE

DIAMETERS, AXES, AND ALGEBRAIC EQUATIONS OF CURVES OF SECOND ORDER

73. Diameters and Centers of Conics 94
74. Conjugate Diameters 94
75. Application of the Harmonic Properties of Poles and Polar Lines 95
76. The Axes of a Conic 97
77. The Vertices of a Conic 99
78. Algebraic Equations of the Conics 100
79. Diametral Planes and Axes of Cylinders 104

CHAPTER X

RULED SURFACES OF SECOND ORDER

80. Ruled Surfaces—Theorem XII 105
81. Sections of a Surface of Second Order 106
82. Tangent Lines and Tangent Planes 107
83. Tangent Cones 108
84. Polar Planes 108
85. Circumscribing Tetrahedrons 109
86. The Class of a Ruled Surface of Second Order 109
87. Classification of Ruled Surface of Second Order 109

CHAPTER XI

PROJECTIVELY RELATED ELEMENTARY FORMS

88. Four Harmonic Elements of an Elementary Form 112
89. Elementary Forms in Perspective Positions with Primitive Forms 113
90. Elementary Forms in Perspective Positions with Each Other 114
91. Projectively Related Forms 114
92. Determination of Projective Relationship 115
93. Superposition of Projectively Related Forms 116
94. Double Elements of a Projectivity 116
95. The Axis of a Projectivity on a Conic 117
96. The Center of a Projectivity on a Conic 118
97. Double Elements of a Projectivity on any Form 119

PAGE
98. Application of the Theorem in Art. 95 119
99. Classification of Projectivities on a Form 120
100. Cyclic Projectivities 121
101. Construction of Cyclic Projectivities 121

CHAPTER XII

THE THEORY OF INVOLUTION—IMAGINARY ELEMENTS

102. Definition of Involution 127
103. Fundamental Theorems—Theorem XIII. 127
104. Hyperbolic Involutions 128
105. Elliptic Involutions. 128
106. Parabolic Involutions. 129
107. Involutions on a Straight Line 130
108. Involutions on a Sheaf of Rays of First Class. 132
109. Involutions Determined by a Complete Quadrangle, or a Complete Quadrilateral 133
110. Desargues Theorem and Its Dual 134
111. Involutions Determined by a Fixed Conic. 135
112. Imaginary Points on a Straight Line 137
113. Imaginary Lines in a Plane 138
114. Imaginary Planes 139
115. Construction Problems 139
116. Imaginary Elements on any Form 141
117. Construction Problems 141

CHAPTER XIII

THE FOCI AND FOCAL PROPERTIES OF CONICS

118. Definition of Focus. 146
119. Construction of Foci 146
120. Directrices and Focal Radii 149
121. Fundamental Theorem—Theorem XIV 150
122. Consequences of Theorem XIV 150

CHAPTER XIV

PROJECTIVELY RELATED PRIMITIVE FORMS OF THE SECOND KIND

123. Primitive Forms of the Second Kind 159

PAGE
124. Perspective Position of Planes and Bundles 159
125. Orthogonally Correlated Bundles. 160
126. Definition of Projective Relationship 160
127. Definition of Collineation 161
128. Definition of Duality 161
129. Consequences of the Foregoing Definitions 161
130. Fundamental Theorem—Theorem XV 161
131. Determination of Projective Relationship 163
132. Projectivities in a Form of Second Kind. 165
133. The Perspectivity 165
134. Construction of a Perspectivity in a Plane. 166
135. The Invariant of a Perspectivity in a Plane 167
136. The Harmonic Perspectivity or Involution in a Plane. . . 168
137. Limiting Lines in a Collineation 168
138. The Affinity 170
139. Fundamental Property of an Affinity 170
140. Corresponding Conics in Affinately Related Planes 172
141. The Area of a Parabolic Segment. 173
142. The Theorem of Appolonius 173
143. The Similitude. 176
144. Properties of the Similitude 176
145. Inverse Points. Radical Axis 177
146. The Congruence 178
147. Collineation in the Plane. Self-corresponding Elements. 179

CHAPTER XV

POLARITIES IN A PLANE AND IN A BUNDLE

148. The Polarity in a Plane. 183
149. Construction of a Polarity in a Plane 184
150. Self-conjugate Points in a Polarity 185
151. Classification of Polarities in a Plane 187
152. The Polarity in a Bundle 189
153. The Orthogonal Polarity 190
154. Polarity and Anti-polarity with Respect to a Circle. . . . 190
155. The Absolute Polarity 192
156. Double Polarities in a Plane and in a Bundle 192
157. Common Elements in a Double Polarity 193
158. Double Conjugate Elements 194
159. Confocal Elements in a Double Polarity. 196

PAGE
160. Construction of the Confocal Elements of a Double Polarity 196
161. Application to Cones of the Second Order 199
162. Cyclic Planes and Focal Axes of Cones 201
163. Quadric Transformations 202
164. Perspective Quadric Transformations 203
165. Inversion with Respect to a Circle 204
166. Properties of Positive Inversions 205
167. Circular Transformations 207
168. General Note 208

INDEX. 211

PROJECTIVE GEOMETRY

CHAPTER I

THE ELEMENTS AND THE PRIMITIVE FORMS

PROJECTION AND SECTION

1. Central Projection.—If any object, for example a flat geometric figure, is projected from the eye and a plane is intercepted between the eye and the object, there will be marked out on this plane a new figure which bears certain resemblances to the original one. The figure may be changed in size and shape, but the relative positions of lines and points to each other will remain unaltered. If two lines intersect in the original figure, the corresponding lines will intersect in the new. If two points are joined by a line in the original figure, the corresponding points will lie on the corresponding line in the new. The process of obtaining the new figure from the original figure is called **central projection.**

Projective geometry deals primarily with those properties of figures which are unaltered by central projection. Such properties may be studied by analytical methods in which an algebraic symbolism is used; or, as in this book, the methods of pure geometry may be followed, in which the geometric concept is kept constantly in mind. Projective pure geometry, because of its methods, is sometimes called Modern Synthetic Geometry, a name which distinguishes it from the ancient elementary pure geometry on the one hand, and the more recent analytical geometry on the other. Because it is concerned with the relative positions of the parts of a figure and not at all with their magnitudes, the projective pure geometry, or modern synthetic geometry, is known also as the Geometry of Position.

2. The Elements.—The point, the line, and the plane are the **elements** of projective geometry. These elements are undefined and each of them may be thought of as existing independently of

the others; that is, the line may be thought of as existing without regard to the points lying on it, and the plane may be considered apart from the points and lines lying on it.

The line, by which we shall always mean straight line, or ray, and the plane are regarded as unlimited in extent.

3. Notation.—To represent the elements we shall adopt as notation, italic capital letters, A, B, C,—to denote points; small italics, a, b, c,—to denote lines or rays; and small Greek letters, α, β, γ,—to denote planes. The line through two points A and B will be denoted by AB; the line of intersection of two planes α and β will be denoted by $\alpha\beta$; the plane determined by the point A and the line b will be denoted by Ab; the point determined by the line a and the plane β will be denoted by $a\beta$ and so on.

4. Primitive Forms.—Each element may be regarded as the **base** or **support** of an indefinite number of elements of another kind. For example, a plane may be regarded as the support of the indefinite number of points and lines which lie on it; a line may be thought of as the support of the indefinite number of planes which can be drawn through it, or of the indefinite number of points which can be placed on it. Again, through a point may be drawn an indefinite number of planes or lines. Such combinations of elements are known as **primitive forms.** They are defined as follows:

1. The **point-row** or **range of points** is the aggregate of points lying on a straight line. The straight line is the base or support of the point-row and is unlimited in extent.

2. **The sheaf of lines, or rays,** is the aggregate of lines or rays lying in one plane and passing through one point. The point is the center or support of the sheaf.

3. **The sheaf of planes** is the aggregate of planes passing through one line. The line is the axis of the sheaf.

4. **The field of points** is the totality of points lying in one plane. The plane is the support of the field.

5. **The field of lines, or rays,** is the totality of lines, or rays, lying in one plane. The plane is again the support of the field.

6. **The bundle of lines, or rays,** is the totality of lines or rays passing through one point. The point is the center or support of the bundle.

7. **The bundle of planes** is the aggregate of planes passing through one point. The point is again the center or support of the bundle.

8. All points in space constitute a primitive form called the **space of points**; all planes in space form the **space of planes**; all the lines in space meeting a given line form a **special linear complex of rays**; and all lines, or rays, in space form the **space of lines, or rays.**

There are, then, in all eleven distinct primitive forms with which we have to deal.

5. Projection and Section.—Projection and section are the common processes of projective geometry. When we look at any object, a building for example, every visible point determines with the eye a ray which is called the **projector** of the point. Every visible straight line of the building determines with the eye a plane which is the projector of the line. The bundle of which the eye is the center and which is composed of the projectors of all visible points and lines of the building is called the projector of the building. The projector of a curved line on the building is a conical surface composed of the projectors of all the points of the curve. This conical surface is part of the bundle which is the projector of the entire building.

If we interpose a plane between the eye and the building, and imagine that each projector makes its passage through the plane visible, we shall have a detailed picture of the building on the plane. This picture is a **section** of the bundle. The section of an individual projector by the plane is called the **trace** of that projector on the plane and the section of the bundle is made up of the traces of all the projectors on the plane. The picture or diagram so obtained is called a **projection** of the building on the plane. In general, the projector of a point from a point is a ray, and of a ray is a plane. The projection of a point on a plane is a point, and of a line is a line.

The process of projection and section is familiar in photography. Here the projectors of the various points and lines of the object photographed, drawn to the lens of the camera, are produced back of the lens and leave their traces on the sensitized plate. The photograph obtained is a section of the bundle of projectors and a projection of the object photographed. Again, the retinal image

in the eye is a section of the bundle of projectors from the landscape at which we are looking, the center of the bundle being the eye. The retinal image is a projection of the landscape.

The draftsman makes use of central projection and section when he constructs the elevation of a dwelling. Such an elevation is but a section of the rays drawn from an assumed position of the observer to all the visible points of the dwelling; that is, it is a projection of the dwelling.

6. Projection and Section Applied to the Primitive Forms.— Let u be any point-row containing the points $A, B, C, D, \ldots$, and let S be any point not on this point-row. Connect S with each of the points $A, B, C, D, \ldots$. These lines are the pro-

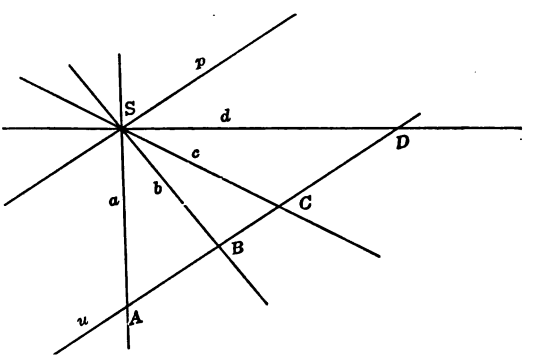


FIG. 1.

jectors of the points of the point-row from the center S, and when each is produced indefinitely in both directions we obtain a sheaf of rays whose center is S. Hence the projector of a point-row, from a point not lying on it, is a sheaf of rays.

Similarly, if we project a sheaf of rays whose center is S, from a point not lying in its plane, from the eye for example, we obtain a sheaf of planes whose axis is the projector of the center S.

If we project a field of points, or a field of lines, from a point not lying in the field, we obtain a bundle of rays, or a bundle of planes.

Again, if we cut a sheaf of rays by a line not passing through

its center we obtain a point-row, and if we cut a sheaf of planes by a plane not passing through its axis we obtain a sheaf of rays.

In the same way, the section of a bundle of rays is a field of points and the section of a bundle of planes is a field of lines. The primitive forms can thus be derived, one from another, by means of projection and section.

Any plane figure consisting of points and lines can be projected from a point S not lying in the plane. The projector thus obtained consists of rays and planes passing through S. This can be cut by any plane not passing through S, and a second plane figure is obtained. We say that, by this process, the first figure has been projected into the second, or that the second figure is a projection of the first.

7. Ideal Elements.—In Fig. 1, suppose that a ray, unlimited in extent, rotates about S in counterclockwise direction, passing in turn through the positions $a, b, c, d, \ldots$. Its intersection with the line u will move along u always in the same direction, passing in turn through the positions $A, B, C, D, \ldots$. As the rotation continues, the point of intersection of this ray with u is, for the instant, lost to view far to the right and immediately thereafter it appears far to the left still moving along u in the same direction as before. It is assumed that for one and only one position of the rotating ray, the intersection with u is lost to the senses, or is not in the finite region. In other words, it is assumed that on the straight line u there is one and only one infinitely distant point, and that this point makes the line continuous from extreme right to extreme left, or *vice versa*. This point is called the **ideal point** on the line, other points on the line being called **actual points.** The ideal point is commonly called the infinitely distant point on the line.

That ray of the sheaf S which passes through the ideal point of u is said to be **parallel** to u. Through S there is thus one and only one line parallel to u.

What has been said about the line u obviously holds for any other straight line, hence we have the **fundamental assumption:** *On every straight line there is one and only one ideal or infinitely distant point. This point makes the line continuous from any one point on it to any other point on it in either direction. Through a*

given point there can be drawn one and only one line parallel to a given line. This parallel intersects the given line in the ideal or infinitely distant point.

Suppose that in a plane a line a rotates about a fixed point S. Every point of a, including the ideal point, traces out a path. The rotating line will eventually be parallel to any given line in the plane and, in this position, its ideal point will coincide with the ideal point on the given line. The ideal point on a will therefore eventually coincide with each ideal point in the plane and hence the path of the ideal point on a includes all the ideal points in the plane. This locus has the property of being met by any line of the plane in but one point, and hence it is itself a straight line. This straight line, the locus of all the ideal points in the plane, is called the **ideal line,** or the **infinitely distant line** of the plane. Two parallel planes intersect in, or have in common, the same ideal line.

Similarly, the locus of all the ideal points in space is a plane, for if it were a curved surface, some lines, not wholly ideal, would have more than one point in common with it; that is, would have more than one ideal point; and some planes would cut this surface in a curved line and thus have a curved locus of ideal points. This plane, the locus of all the ideal points in space, is the **ideal plane** of space.

A series of parallel lines in a plane is a sheaf of rays whose center is an ideal point. A series of parallel lines in space is a bundle of rays whose center is an infinitely distant point. A series of parallel planes is a sheaf of planes whose axis is an ideal line.

The introduction into geometry of the ideal elements makes it possible to include in a general statement many conditions which otherwise would appear as exceptional cases. For example, any two straight lines in a plane intersect in a point. The point may be actual or ideal. Except for the recognition of ideal points, this statement would read: Any two straight lines in a plane intersect in a point unless they are parallel. Similarly, any two planes intersect in a straight line which may be either actual or ideal. If the line of intersection is ideal, it contains all the ideal points of either plane, and the planes are parallel.

Exercises

1. Project any two lines in a plane into two lines which are parallel to each other.

2. Project a sheaf of lines into a system of parallel lines.

3. Project an arbitrary quadrangle $ABCD$ into a parallelogram.

Suggestions.—Let AD, BC meet in E and AB, DC in F. Let S be the point from which the projection is made. Cut the quadrangular pyramid S-$ABCD$ by a plane parallel to the plane SEF.

4. Project any two lines of a plane into two perpendicular lines.

Suggestions.—Let a and b be the lines and A and B be two points on a and b respectively. Pass a second plane through AB and in it construct a right triangle on AB as hypothenuse. Let S be the right angle. Project from S and cut by a plane parallel to SAB.

5. Project any plane quadrangle into a rectangle.

6. Project any plane quadrangle into a square.

Suggestion.—A rectangle whose diagonals are perpendicular to each other is a square.

7. Project any triangle into an equilateral triangle.

Suggestion.—Pass a second plane through one of the sides and in it construct an equilateral triangle on that side as base. Project from the vertex.

8. What plane figure results from cutting a tetrahedron by an arbitrary plane?

9. How can one cut a cube so as to obtain a regular hexagon?

CHAPTER II

THE PRINCIPLE OF DUALITY—SIMPLE AND COMPLETE FIGURES

8. Duality in the Plane.—We have already noticed (Art. 7) that the recognition of the ideal points in a plane makes it possible to say, without exception:

1. *Two lines a and b in a plane intersect, and so determine, the point ab.*

Also in this plane:

2. *Two points A and B are joined by, and so determine, the line AB.*

Comparing statement (1) with statement (2), we see that either may be derived from the other by a simple interchange of point with line. The two statements are said to be **plane-duals,** or **reciprocals,** of each other. The elements point and line are **dual elements,** or **reciprocal elements,** in the plane.

The interchange of dual elements may be illustrated by writing one statement over the other, thus:

Two $\begin{bmatrix} \text{lines } a \text{ and } b \\ \text{points } A \text{ and } B \end{bmatrix}$ determine the $\begin{bmatrix} \text{point } ab \\ \text{line } AB \end{bmatrix}$.

In the same way, from any geometrical figure consisting of points and lines in a plane, can be derived the dual, or reciprocal, figure by interchanging point with line. The following are examples.

1. All the $\begin{bmatrix} \text{points on a line} \\ \text{lines through a point} \end{bmatrix}$ constitute a $\begin{bmatrix} \text{point-row} \\ \text{sheaf of rays} \end{bmatrix}$.

2. All the $\begin{bmatrix} \text{points} \\ \text{lines} \end{bmatrix}$ in a plane constitute a field of $\begin{bmatrix} \text{points} \\ \text{lines} \end{bmatrix}$.

3. Three $\begin{bmatrix} \text{points} \\ \text{lines} \end{bmatrix}$ not $\begin{bmatrix} \text{lying on a line} \\ \text{passing through a point} \end{bmatrix}$ are the $\begin{bmatrix} \text{vertices} \\ \text{sides} \end{bmatrix}$ of a triangle.

4. A moving $\begin{bmatrix} \text{point describes} \\ \text{line envelopes} \end{bmatrix}$ a curve.

Similarly, any theorem involving only the relative positions of points and lines in a plane has a dual, or reciprocal, theorem involving the relative positions of the dual elements, line and point.

Further examples of the use of duality in the plane will appear as we proceed.

9. Duality in Space.—In space the following statements hold.

1. Three [planes / points] not [passing through / lying on] the same [line / line] determine a [point / plane].
2. Two [points A and B / planes α and β] determine the [line AB / line $\alpha\beta$].
3. A [point / plane] and a [line / line] determine a [plane / point].
4. Two [lines / lines] which have a common [point / plane] determine a [plane / point].

A study of these statements shows that the point and the plane are dual elements in space, while the line is its own dual, or is self-dual.

Again,

5. All the [points on a line / planes through a line] constitute a [point-row / sheaf of planes].
6. All the [lines passing through a point / lines lying in a plane] constitute a [bundle / field of lines / of lines].
7. All the [points lying in a plane / planes passing through a point] constitute a [field / bundle of points / of planes].

From these examples it will be seen that each primitive form has a space-dual, or reciprocal, primitive form. The space of points and the space of planes are reciprocal forms. The space of lines is self-dual. Also the special linear complex of rays is self-dual in space.

The following is an example of a theorem and its reciprocal placed in parallel columns.

If four points A, B, C, D, are so situated that the lines AB and CD intersect, then all the points lie in one plane and consequently the lines AC and BD, and also the lines AD and BC, intersect.

If four planes $\alpha, \beta, \gamma, \delta$, are so situated that the lines $\alpha\beta$ and $\gamma\delta$ intersect, then all the planes pass through one point and consequently the lines $\alpha\gamma$ and $\beta\delta$, and also the lines $\alpha\delta$ and $\beta\gamma$, intersect.

10. The Principle of Duality.—The principle of duality asserts that a dual, or reciprocal, statement can be derived from a given statement as in the foregoing examples. It may be formulated as follows:

The principle of duality asserts that from any statement or theorem concerning the relative positions of the elements composing a geometrical configuration, another statement or theorem can be obtained by a simple interchange of the elements of the configuration with their reciprocals.

If, for example, the original theorem asserts that three lines in a plane meet in a point, then the reciprocal, or dual, theorem in the plane asserts that three points lie on a line.

Again, if the original theorem states that four planes pass through a point, the reciprocal theorem asserts that four points lie in a plane.

Exercises

1. What is the plane-dual of a triangle? The space-dual?

2. Show that the dual of a tetrahedron is another tetrahedron; that the hexahedron and octahedron are dual figures; that the dodecahedron and icosahedron are dual figures.

3. What plane figure is obtained by cutting a cube with a plane not passing through any vertex? By projecting a tetrahedron from any point not in a face?

4. If the points A, E, C lie on a line a and the points D, B, F on a second line b, then the lines, AB, ED; AF, DC; and EF, BC intersect in three points K, L, and M which lie on a third line c. Write out carefully the reciprocal theorem in the plane and draw the corresponding figure. Also write out the space-dual of the given theorem.

11. Simple Figures in a Plane.—Besides the point-row and the sheaf of rays, the polygons of elementary geometry form the

simplest combinations of points and lines in a plane. A **simple polygon** consists of a number of points (vertices) arranged in some definite order and joined two and two in this order by straight lines (sides), the last point being joined to the first. The polygon may be convex or reentrant, depending upon the arrangement of the vertices chosen. For example, from the four points, A, B, C, D (Fig. 2) can be formed the three simple quadrangles $ABCD$, $ACDB$, and $ADBC$. Two of these quadrangles are reentrant, namely, $ACDB$, and $ADBC$.

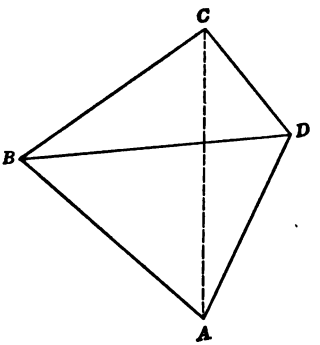


FIG. 2.

A simple polygon has as many sides as vertices and its dual in the plane is another simple polygon having an equal number of sides and vertices.

A simple polygon having n vertices, and consequently n sides, is often called a **simple n-point** or a **simple n-side.** A simple n-point has $2n$ elements, namely, the n vertices and the n sides.

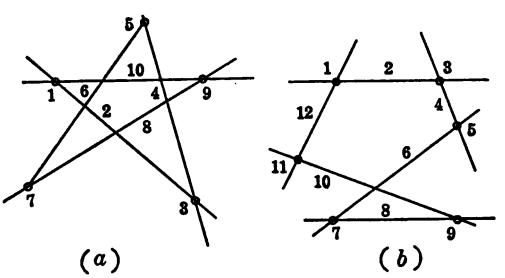


FIG. 3.

Corresponding to any element there is always an opposite element. For example, in the simple 5-point, shown at (a), Fig. 3, the side 8 is opposite the vertex 3, the side 10 is opposite the vertex 5, and so on. In the simple 6-point, shown at (b), the vertex 3

is opposite the vertex 9, the side 4 is opposite the side 10, and so on. In general, if the elements of any simple n-point are numbered in order, as in Fig. 3, and if $k \leq n$, then the elements k and $k + n$ are opposite each other. If $k > n$, then k and $k - n$ are opposite elements.

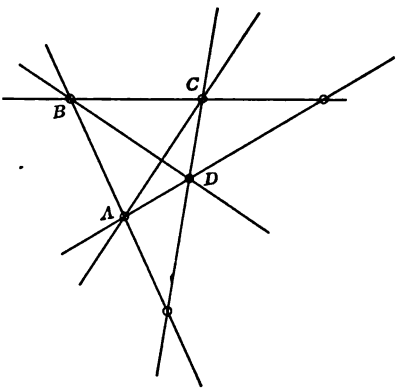


FIG. 4.

12. Complete Figures in a Plane.—If n points in a plane are joined two and two in all possible ways, a figure is obtained called

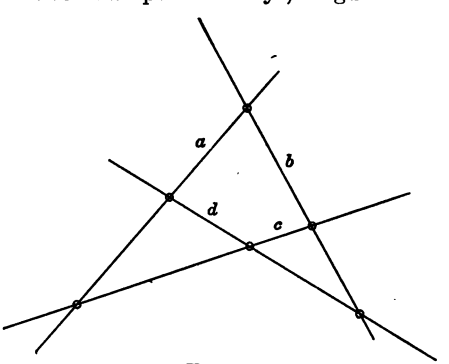


FIG. 5.

a **complete n-point.** For example, if the four points A, B, C, D (Fig. 4) are joined in pairs in all possible ways, we obtain the six lines AB, AC, AD, BC, BD, and CD. The figure is a complete 4-point, or a **complete quadrangle.**

Again, if n lines in a plane intersect two and two in all possible ways, the figure obtained is called a **complete n-side.** Thus, in Fig. 5, the four lines a, b, c, d intersect in pairs and so determine the six points ab, ac, ad, bc, bd, and cd. The figure is a complete 4-side, or **complete quadrilateral.** The complete quadrangle and the complete quadrilateral are dual figures in the plane.

A complete 5-point, or complete pentagon, consists of 5 points and the 10 lines joining the points two and two. The dual figure is a complete 5-side, or complete pentalateral, and consists of 5 lines together with their 10 points of intersection, two and two.

In general,

A complete n-point consists of n points in the plane together with all the $\frac{n(n-1)}{2}$ lines joining the points, two and two.	A complete n-side consists of n lines in the plane together with all the $\frac{n(n-1)}{2}$ points of intersection of the lines, two and two.

13. Simple and Complete Figures in Space.—The projector of a simple, or a complete, plane figure from a point not in its plane, is a simple, or complete, figure in space. Thus, the projector of a simple n-point is a **simple n-edge;** that is, an ordinary pyramid of elementary geometry having n edges and n faces. The projector of a complete n-point is a **complete n-edge** having n edges and $\frac{n(n-1)}{2}$ faces; and the projector of a complete n-side is a **complete n-face** having n faces and $\frac{n(n-1)}{2}$ edges. A simple or a complete n-edge, or n-face, thus consists of a combination of lines and planes passing through a point, and is, therefore, contained within a bundle of lines or a bundle of planes.

The ordinary polyhedrons of elementary geometry are simple combinations of points, lines, and planes. Thus an octahedron has 6 points (vertices), 12 lines (edges), and 8 planes (faces). An easy generalization leads to complete combinations of points, lines, and planes. Thus a complete 5-point in space (Fig. 6) consists of 5 points (vertices) together with the 10 lines (edges) joining the points two and two and the 10 planes (faces) de-

termined by taking the points in threes. The reciprocal figure is a complete 5-plane consisting of 5 planes (faces) together with the 10 lines (edges) determined by taking the planes in pairs and the 10 points (vertices) determined by taking the planes in threes.

In general,

A complete n-point in space has $\frac{n(n-1)}{2}$ edges and $\frac{n(n-1)(n-2)}{6}$ faces.

A complete n-plane in space has $\frac{n(n-1)}{2}$ edges and $\frac{n(n-1)(n-2)}{6}$ vertices.

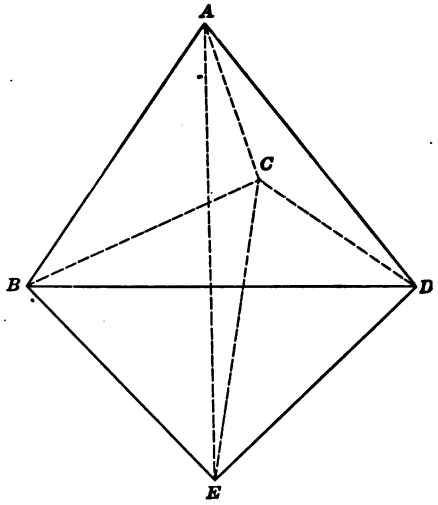


FIG. 6.

Many interesting and important combinations of points, lines, and planes arise from a study of the simple and complete figures defined in this and the preceding articles. As an example, through each edge of a complete 5-point in space there pass three faces and in each face there lie three edges. Thus, through the edge AD (Fig. 6) there pass the faces BAD, EAD, and CAD; and in the face ABC lie the three edges AB, AC, and BC. Hence, if we cut the 5-point by a plane not passing through any vertex, we obtain

a plane figure consisting of 10 lines and 10 points which are respectively the traces of the 10 faces and the 10 edges. In this plane figure, three lines pass through each point and three points lie on each line. This combination of points and lines in a plane, thus shown to be possible, is known as the **configuration of Desargues.** It is of great importance in the development of projective geometry from the point of view we have chosen.

Exercises

1. Construct a triangle having one vertex at infinity; having two vertices at infinity.

2. $ABCDE$ is a simple pentagon, D and E being ideal points. Construct the figure and point out the vertex opposite the infinitely distant side; the sides opposite the ideal vertices.

3. Construct a complete 5-point in the plane, two vertices being at infinity.

4. Given any plane figure, show that the space-dual of this figure is a projector of its plane-dual.

5. Construct a complete 6-point in space. Describe a section of this 6-point by a plane not passing through any vertex. Describe the reciprocal configuration in the plane.

6. Show that the plane-dual of a configuration of Desargues is another configuration of Desargues.

7. Construct a complete 5-plane in space. What is a section of this figure by a plane not passing through any vertex?

NOTE.—The principle of duality is due to two French mathematicians, Poncelet (1822) and Gergonne (1826). Desargues (1593–1662) was one of the founders of modern geometry.

CHAPTER III

CORRELATION OF GEOMETRIC FIGURES
PERSPECTIVE POSITION OF GEOMETRIC FIGURES

14. Correlation of Geometric Figures.—Any combination of the elements is a **geometric figure.** Primitive forms, simple figures, and complete figures are thus geometric figures. The points, lines, and planes of a geometric figure are its **elements.** Two elements of a geometric figure that lie on, or pass through, each other are called **incident elements.** For example, if the point A lies on the line a, the elements A and a are incident. Again, if the plane α passes through the line a, a and α are incident elements. Each point of a geometric figure is thus incident to all the lines and planes of the figure passing through it; each line is incident to all the points of the figure lying on it and to all the planes of the figure passing through it; and each plane is incident to all the points and lines of the figure lying in it.

If two geometric figures are so related that to any element of either corresponds a definite element of the other and, further, if incident elements in either always correspond to incident elements in the other, then the figures are said to be **correlated one to one.**

Corresponding elements in correlated figures are often called **homologous elements.**

As an example; if in two correlated quadrangles, the vertices A, B, C, D of one correspond respectively to the vertices A_1, B_1, C_1, D_1 of the other, then any side, as AB, of the one corresponds to the side A_1B_1 of the other, since AB is incident to both A and B and must correspond, by definition, to the side A_1B_1, incident to both A_1 and B_1. Similarly, AC, A_1C_1; AD, A_1D_1; etc., are pairs of homologous sides.

15. Perspective Position of Geometric Figures.—Two primitive forms are said to be **unlike** if one is formed of elements differing in kind from the elements composing the other. Thus a

point-row and a sheaf of lines are unlike primitive forms. Like primitive forms are each composed of the same kind of elements. Two bundles of rays are like primitive forms.

Two unlike primitive forms are **in perspective position** with each other if one is a section of the other. Any element of the one form then lies on, or passes through, its homologous element of the other. Thus a field of points and a bundle of rays are in

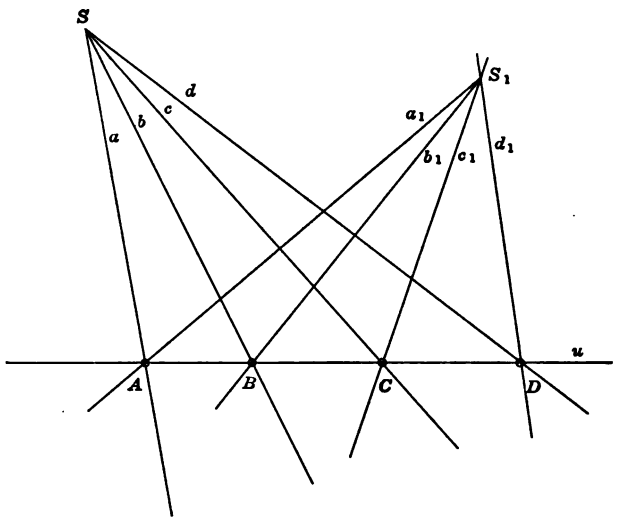


FIG. 7.

perspective position when the first is a section of the second, each point of the field lying on its homologous ray of the bundle.

Two like primitive forms are **in perspective position** if each is a section of, or a projector of, the same third primitive form. Corresponding elements have in common an element of the third primitive form. For example, two sheaves of rays are in perspective position if they are each projectors of the same point-row, and then corresponding rays have in common a point of the point-row (Fig. 7). Two sheaves of rays are also in perspective position if each is a section of the same sheaf of planes, and then corresponding rays have in common a plane of the sheaf of

planes. Since the planes in which lie the sheaves of rays meet in a line, the sheaves of rays are projectors of the point-row cut from this line by the sheaf of planes (Fig. 8).

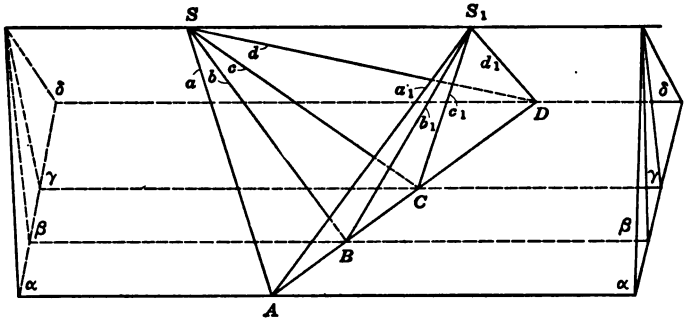


FIG. 8.

Two point-rows are in perspective position if each is a section of the same sheaf of rays, corresponding points then have in

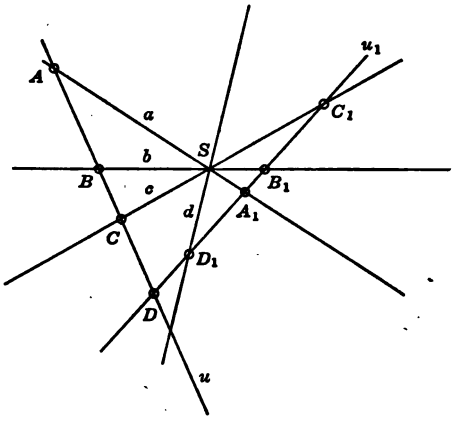


FIG. 9.

common a ray of the sheaf (Fig. 9). Two non-intersecting point-rows are in perspective position if each is a section of the same sheaf of planes, corresponding points then have in common a plane of the sheaf.

Clearly it is not always possible for two primitive forms to be in perspective position. Thus a sheaf of rays and a field of points cannot be in perspective position, for the one cannot be a projector of the other. Again, a point-row and a field of points cannot be sections of any third primitive form. When two primitive forms can be in perspective position, they are said to have **the same dimensions.** Thus, the first three primitive forms (Art. 4) are called primitive forms of **one dimension.** Each can be in perspective position with either of the other two, and cannot be in perspective position with any of the other primitive forms. The primitive forms 4, 5, 6, 7 (Art. 4) are **two-dimensional** primitive forms; the space of points, the space of planes, and the special linear complex are **three dimensional** primitive forms; and the space of rays has **four dimensions.**

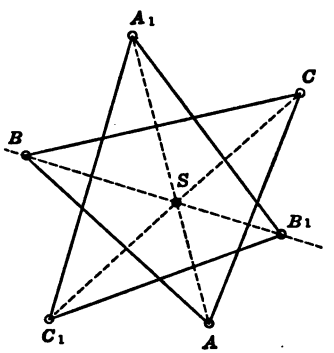


FIG. 10.

Two correlated polygons are in perspective position if the lines joining corresponding vertices meet in a point. For example, two triangles are in perspective position if each is a section of the same 3-edge. If the triangles lie in the same plane, they are in perspective position when the lines joining corresponding vertices meet in a point (Fig. 10).

16. Notation.—The symbol $\overline{\overline{\wedge}}$ is used to designate "is in perspective position with." Thus:

$$\text{The sheaf of rays } S \overline{\overline{\wedge}} \text{ the point-row } u,$$

is read, "the sheaf of rays S is in perspective position with the point-row u," and means that u is a section of S. Again:

$$\text{The triangle } ABC \overline{\overline{\wedge}} \text{ the triangle } A_1B_1C_1,$$

is read, "the triangle ABC is in perspective position with the triangle $A_1B_1C_1$," and means that the lines joining corresponding

vertices meet in a point. When no ambiguity can occur, these statements may be abbreviated thus,

$$S \overline{\overline{\wedge}} u, \text{ and } ABC \overline{\overline{\wedge}} A_1B_1C_1.$$

Exercises

1. If $u \overline{\overline{\wedge}} S$, what is the reciprocal figure in space?

2. ABC and $A_1B_1C_1$ are two correlated triangles in the same plane and in perspective position. Construct the figure. What is the dual figure in the plane? In space?

3. If α and β are two fields of points and $\alpha \overline{\overline{\wedge}} \beta$, what can be said of lines joining corresponding points? Describe the dual figure.

4. Under what condition will two correlated fields of rays be in perspective position?

5. If S and S' are two correlated sheaves of rays and $S \overline{\overline{\wedge}} S'$, what is the locus of the intersection of corresponding rays?

6. Two non-intersecting point-rows are in perspective position. Describe the dual figure.

17. Theorem I.—Desargues Theorem.—*If two correlated triangles are so situated that corresponding sides meet in three points of a straight line, then the triangles are in perspective position.*

Let the vertices A, B, C of one triangle correspond respectively to the vertices A_1, B_1, C_1 of the other; and corresponding sides meet in the points D, E, F of the line u.

If the triangles lie in different planes (Fig. 11), a pair of corresponding sides, as AB and A_1B_1, determines a plane. The three planes so determined are the faces of a triangular pyramid whose edges are the lines joining corresponding vertices. The triangles are thus sections of the same 3-edge and consequently in perspective position.

If the triangles lie in the same plane (Fig. 12), pass a second plane through the line u and in it construct a third triangle $A_2B_2C_2$ so that the sides B_2C_2, A_2C_2, A_2B_2 pass through D, E, F, respectively. Then $ABC \overline{\overline{\wedge}} A_2B_2C_2$, the lines joining corresponding vertices meeting in O; and $A_1B_1C_1 \overline{\overline{\wedge}} A_2B_2C_2$, the lines joining corresponding vertices meeting in O'. The three planes A_2A_1A, B_2B_1B, and C_2C_1C form three planes of the sheaf of planes whose axis is OO', and intersect the plane of the original triangles in the lines AA_1, BB_1, and CC_1, respectively. These lines must, there-

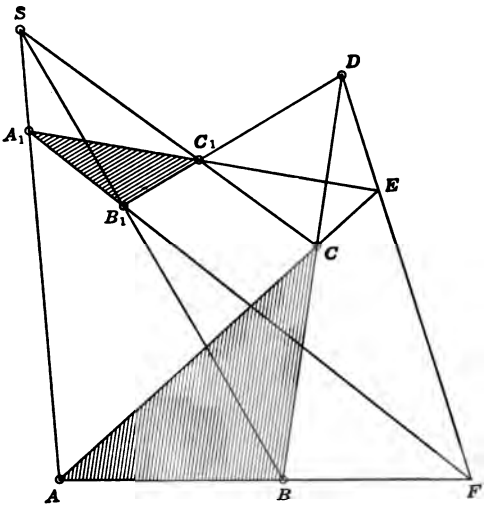


FIG. 11.

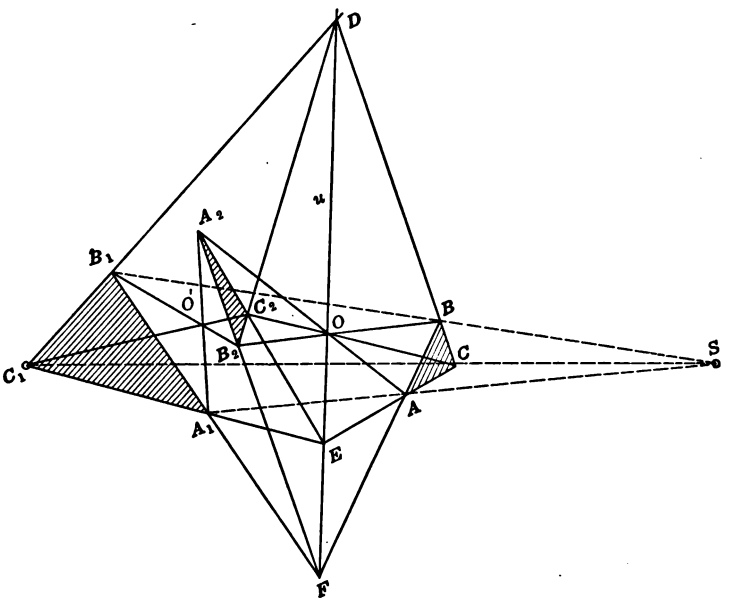


FIG 12.

fore, pass through the point where the axis OO' meets the plane ABC. Hence $ABC \overline{\wedge} A_1B_1C_1$.

18. Converse of Theorem I.—*If two correlated triangles are in perspcetive position, then corresponding sides meet in three points of a straight line.*

If the triangles do not lie in the same plane, they are sections of the same 3-edge and corresponding sides must meet in points of the line common to their planes (Fig. 11).

If the triangles lie in the same plane, let the line joining corresponding vertices meet in S (Fig. 12). Through S draw a line not lying in the plane of the triangles, and upon this line choose two points as O and O'. Project ABC from O and $A_1B_1C_1$ from O', thus obtaining two triangular pyramids whose edges intersect in pairs. For example, BO and B_1O' intersect, since the four points B, O, O', B_1 lie in the plane BSO. Similarly, the other pairs of edges meet and we have a third triangle $A_2B_2C_2$ which is in perspective position with each of the given triangles. The sides of $A_2B_2C_2$ must, therefore, meet the corresponding sides of each of the others in points of the same straight line. This line is the intersection of the plane $A_2B_2C_2$ with the plane in which lie the original triangles.

19. Definitions.—If the triangle $ABC \overline{\wedge}$ the triangle $A_1B_1C_1$, lines joining corresponding vertices meet in the **center of perspectivity,** and corresponding sides intersect upon the **axis of perspectivity.** Thus S is the center of perspectivity and u is the axis of perspectivity in Figs. 11 and 12.

Points lying upon the same straight line are **collinear.** Lines passing through the same point are **concurrent.** Planes through the same line are **coaxial.** Figures in the same plane are **coplanar.**

Two coplanar triangles which are in perspective position form a configuration of Desargues (Art. 13). Thus the 10 points and 10 lines lying in the plane ABC (Fig. 12) form the section of the complete 5-point $OA_2B_2C_2O'$ by the plane ABC.

Exercises

1. Join the middle points of the sides of any triangle ABC forming a second triangle, $A_1B_1C_1$. The two triangles are in perspective position. Where is the axis of perspectivity? The center of perspectivity? Does this prove that the medians meet in a point?

2. To join a given point to the inaccessible intersection of two given lines.

SOLUTION.—Let u and u_1 be the lines and A the point. Through a point O, not lying on either line, draw three lines meeting the given lines in BC, RS, and B_1C_1. Connect A with B and C determining upon RS the points M and N respectively. Draw the lines C_1N and B_1M meeting in A_1. The two triangles ABC and $A_1B_1C_1$ are in perspective position. Complete the solution.

3. Given two lines a and b. Through a point P, not lying on either, draw a series of lines forming with a and b a set of quadrangles. Show that the diagonals of these quadrangles all intersect upon the same straight line which also passes through the intersection of a and b.

Suggestion.—The point P is the center of perspectivity for a series of triangles whose axes of perspectivity all coincide in the line sought. This line has been called the *polar line* of the point P with respect to the two lines a and b (Poncelet).

4. If a point P is connected to the vertices of a given triangle ABC, the joining lines meeting the sides opposite in $A_1B_1C_1$, respectively, the two triangles ABC and $A_1B_1C_1$ are in perspective position. Construct the axis of perspectivity. This axis is often called the *polar line* of the point with respect to the triangle ABC. The point P is called the *pole*. Construct the pole of a given line.

5. Show that in the configuration of Desargues *any* line may be taken as an axis of perspectivity. The two triangles and the center of perspectivity will then be uniquely determined. Also that *any* point may be taken as center of perspectivity. The two triangles and the axis of perspectivity will then be uniquely determined.

6. Project the figure in exercise 3 so that the point P and the intersection of a and b project into ideal points. Prove the property stated in the exercise for the new figure by elementary geometry.

20. Complete Quadrangles in Perspective Position.—Theorem II.—*If two correlated complete quadrangles are so situated that five pairs of corresponding sides meet in points of the same straight line, then the sixth pair also meet on this line and the quadrangles are in perspective position.*

Let $ABCD$ and $A_1B_1C_1D_1$ be two quadrangles so situated that the five pairs of corresponding sides, aa_1, bb_1, cc_1, dd_1, and ee_1 intersect upon u (Fig. 13). Then (theorem I), $ABC \overline{\wedge} A_1B_1C_1$ and $BCD \overline{\wedge} B_1C_1D_1$, the centers of perspectivity coinciding in S. The quadrangles are, therefore. in perspective position. Moreover, $ACD \overline{\wedge} A_1C_1D_1$, and consequently, by virtue of the converse of theorem I, AD and A_1D_1 meet on u.

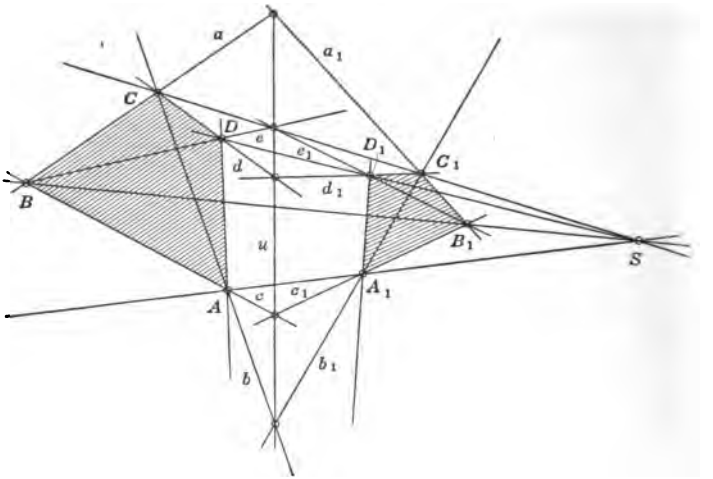


FIG. 13.

Exercises

1. If two complete quadrangles have five pairs of corresponding sides parallel, each to each, show that the sixth pair will be parallel.

2. Given in a plane a fixed parallelogram. By use of the ruler alone, draw a parallel to any given line through a given point.

Suggestions.—Let u be any given line and A a given point. Let u meet the sides of the parallelogram in $MM'NN'$ and the diagonal D_1C_1 in P. Join A to M and M' and through P draw any line cutting these lines in C and D respectively. Join D and C to N and N' respectively and let these lines meet in B. AB is the parallel sought. For the quadrangle $ABCD$ is in perspective position with the quadrangle whose vertices are the two ideal points on the sides of the parallelogram together with the points C_1D_1 (Lambert, 1774).

3. Two triangles are in perspective position but are not coplanar. What is the dual figure?

4. Two complete quadrangles are coplanar and in perspective position. What is the dual figure in the plane?

5. Show that any two complete n-points will be in perspective position provided that the n-2 triangles that can be formed, having two of the vertices of one n-point as common vertices, are in perspective position with their corresponding triangles in the other n-point.

CHAPTER IV

HARMONIC RANGES AND HARMONIC PENCILS

21. Harmonic Ranges.—If four points A, B, C, D are so situated along a line that two sides of a simple quadrangle intersect in A, the two remaining sides intersect in C, one diagonal passes through B, and the other diagonal passes through D, then the four points are called **four harmonic points** and together form **a harmonic range.** The range is said to be **defined** by the quadrangle. Thus (Fig. 14), $ABCD$ is a harmonic range defined by the quadrangle

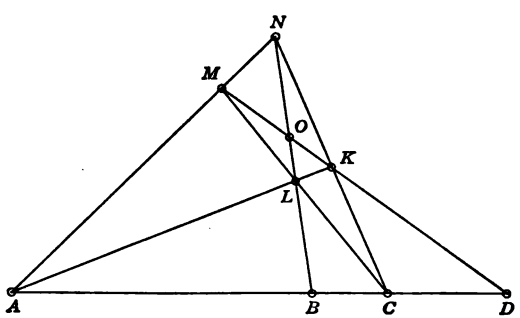


FIG. 14.

$KLMN$, since the sides MN, KL meet in A, the sides ML, NK meet in C, the diagonal NL passes through B, and the diagonal MK passes through D. Also, $MOKD$ is a harmonic range defined by the quadrangle $NALC$, since NA, LC meet in M; AL, NC meet in K; the diagonal NL passes through O; and the diagonal AC passes through D. Similarly, $NOLB$ is a harmonic range defined by the quadrangle $KAMC$.

The points of a harmonic range, taken in order from left to right, are called first, second, third, and fourth harmonic points. The quadrangle which defines the range has two sides meeting

in the first harmonic point, two meeting in the third, one diagonal passing through the second, and one diagonal through the fourth.

22. Effect of Order.—It is clear, from the definitions in the preceding article, that if *ABCD* is a harmonic range, so is *CBAD* and also *ADCB*. In fact, we may interchange the first with the third, or the second with the fourth, or make both these interchanges at the same time and the resulting range will be harmonic and will be defined by the same quadrangle *KLMN* (Fig. 15). In this way *KLMN* defines four harmonic ranges; namely, *ABCD*, *ADCB*, *CBAD*, and *CDAB*. But this is not all, for if the diagonals of the quadrangle intersect in *O*, then the triangles

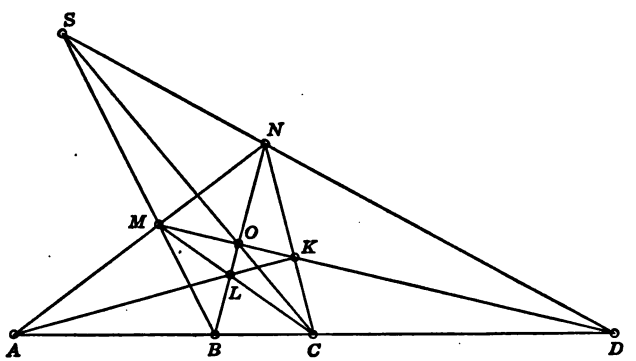


FIG. 15.

OBD and *CMN* are in perspective position since corresponding sides meet on *AK* (theorem I). Hence, the lines *OC*, *BM*, and *DN* meet in a point *S* thus forming a quadrangle *OMSN*. This quadrangle defines the harmonic ranges *DCBA*, *DABC*, *BCDA*, and *BADC*. Hence we can say: *Of the 24 possible arrangements of four points on a line, 8 are harmonic ranges if one is a harmonic range.*

Of these 8 harmonic ranges it is to be noted that *A* and *C* are always separated by *B* and *D*; that is, it is not possible to pass continuously from *A* to *C* in either direction without passing over either *B* or *D*. In none of the remaining 16 possible arrangements are *A* and *C* separated by *B* and *D* as the reader can readily verify.

In any harmonic range $ABCD$, A and C, and also B and D, are called **harmonic conjugates.** One pair of harmonic conjugates always separates the other pair. Either pair is said to be **harmonically separated** by the other.

23. Construction of Harmonic Ranges.—Theorem III.—*Three points of a straight line may be chosen arbitrarily as the first, second, and third of a harmonic range, the fourth harmonic point is then uniquely determined.*

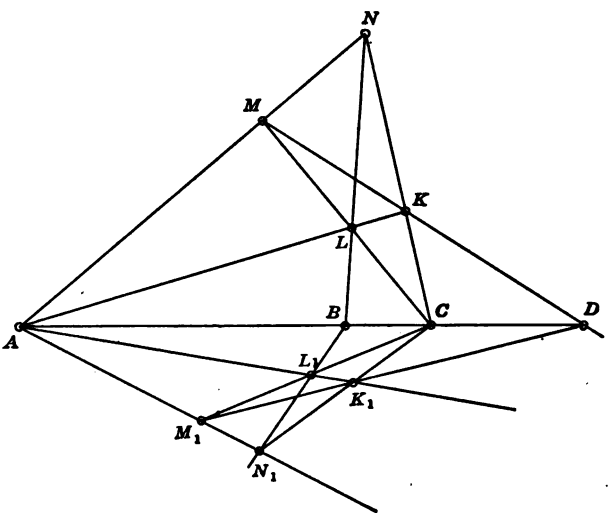


FIG. 16.

For if A, B, C are the points chosen arbitrarily, we can draw any two lines through A and any line through B meeting the lines through A in L and N (Fig. 16). By joining L and N to C a quadrangle $KLMN$ is formed whose diagonal MK determines the fourth harmonic point D. Any other quadrangle as $K_1L_1M_1N_1$, constructed like $KLMN$, will determine the same point D, for

$$KLMN \overline{\overline{\wedge}} K_1L_1M_1N_1,$$

since five pairs of corresponding sides of the complete quadrangles $KLMN$ and $K_1L_1M_1N_1$ meet on the line ABC by construction.

The sixth pair, namely, MK and M_1K_1, must, therefore, meet on this line (theorem II).

24. Harmonic Pencils.—The projector of a harmonic range, from any point not on the line containing the range, is called a **harmonic pencil of rays.** The point common to the rays is the **vertex** of the pencil.

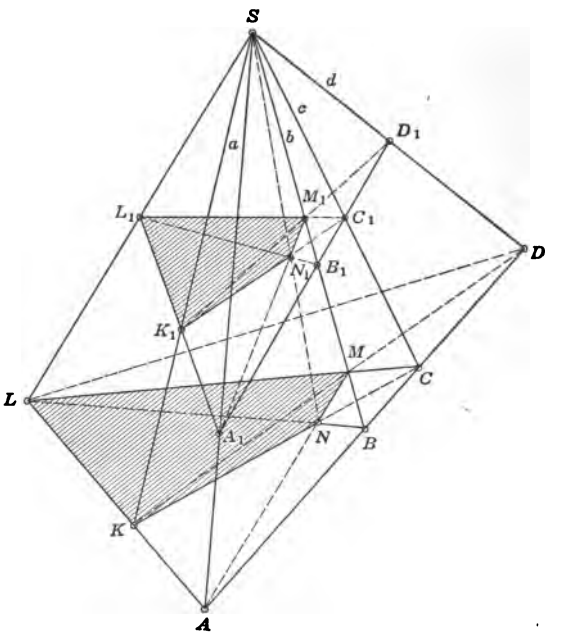


FIG. 17.

The projector of a harmonic pencil of rays, from any point not on the plane of the pencil, is a **harmonic pencil of planes.** The line common to the planes is the **axis** of the pencil.

25. Sections of Harmonic Pencils.—Theorem IV.—*The section of a harmonic pencil of rays by any line not passing through its vertex is a harmonic range of points. The section of a harmonic pencil of planes by any plane not containing its axis is a harmonic pencil of rays.*

Let $ABCD$ be a harmonic range defined by the quadrangle $KLMN$ (Fig. 17). Project the entire figure from a point S not on the plane $KLMN$, thus obtaining a quadrangular pyramid S–$KLMN$ and a harmonic pencil of rays S–$ABCD$. Opposite faces of the pyramid intersect in SA and also in SC; the diagonal planes pass through SB and SD. Hence a section of this figure, by any plane not passing through S, consists of a quadrangle $K_1L_1M_1N_1$ which defines the harmonic range $A_1B_1C_1D_1$. But this range is any section of the harmonic pencil S–$ABCD$.

The pencil of planes SN–$ABCD$ is by definition a harmonic pencil of planes since it is the projector of the harmonic pencil of rays N–$ABCD$ from the point S. Any section of this harmonic pencil of planes is a harmonic pencil of rays, since the cutting plane will meet the pencil N–$ABCD$ in a harmonic range, and the section of the pencil of planes is a projector of this range.

In the harmonic pencil of rays S–$ABCD$, the rays SA and SC are harmonically separated by the rays SB and SD and *vice versa*. Similarly, in the pencil of planes SN–$ABCD$, the planes SNA and SNC are harmonically separated by the planes SNB and SND.

Exercises

1. Given any three rays of a sheaf of rays, construct the fourth harmonic ray.

Suggestion.—Cut the three given rays by any line in the points A, B, C. Use theorem III.

2. Given three planes of a sheaf of planes, construct the fourth harmonic plane.

3. A line rotates about a fixed point A meeting two fixed coplanar lines, b and c, in the variable points B and C respectively. What is the locus of the fourth harmonic point to A, B, C? Consider the case when b and c are parallel.

4. A line passes through a fixed point A and meets two fixed planes in the variable points B and C. What is the locus of the fourth harmonic point?

5. Given three fixed lines a, b, c, no two of which intersect. A fourth line moves so as to meet the fixed lines in the variable points A, B, C respectively. What is the locus of the fourth harmonic point?

Metric Properties of Harmonic Ranges and Harmonic Pencils

26. Harmonic Conjugate of an Ideal Point.—Let D be the ideal point on the point-row AC (Fig. 18). Through A draw any two lines meeting the ideal line in the points K and M. Join K and

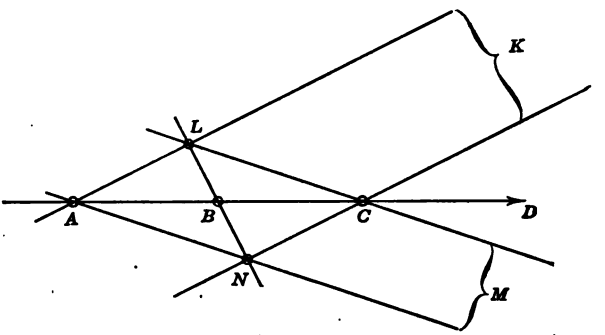


Fig. 18.

M to C, determining upon AK and AM the points L and N respectively. The diagonal LN of the quadrangle $KLMN$ bisects the segment AC, since $LANC$ is a parallelogram. Hence: *If D is the ideal point on the point-row AC, its harmonic conjugate with respect to A and C bisects the segment AC.*

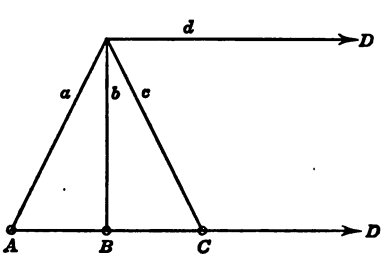


Fig. 19.

27. Normal Conjugate Rays in a Harmonic Pencil.—Let $abcd$ be a harmonic pencil of rays in which the conjugate rays b and d are normal, *i.e.*, perpendicular, to each other (Fig. 19). If we cut this pencil by a line perpendicular to b, and, therefore, parallel to d, we obtain a harmonic range (theorem IV) one of whose points, viz., D, is ideal. Consequently B bisects AC, b bisects the angle (ac), and d bisects the supplement of the angle (ac). Hence:

If two conjugate rays of a harmonic pencil are normal to each other, they bisect the angles formed by the other pair of conjugate rays.

Exercises

1. State and prove the converse of the theorem in Art. 26.

2. State and prove the converse of the theorem in Art. 27.

3. Bisect a given segment AC, having given a parallel to it.

Suggestion.—Project A and C from a point L, determining upon the given parallel the points K and M respectively. Let KC and MA meet in N. The line LN bisects AC.

4. Draw a parallel to a given line AC through a given point K, knowing the middle point of the segment AC.

Suggestion.—Let B be the middle point. Join K to A and C and through B draw any line meeting KA and KC in L and N respectively. Join L to C and N to A determining a point M. The line KM is parallel to AC.

5. In a plane are given a parallelogram and a segment AC of a straight line; it is required, without the use of circles, to bisect AC and to draw a parallel to AC; also to divide AC into n equal parts.

6. Through a given point P draw a straight line meeting two given lines of the plane in A and B so that (1) the segment AB shall be bisected at P, (2) the segment AP shall be bisected at B. Under what circumstances is the solution impossible?

7. A given straight line intersects the sides of a triangle ABC in the points $A_1B_1C_1$. If the harmonic conjugate of each of these points with respect to the two vertices on the same side is joined to the opposite vertex, show that the three lines thus obtained meet in a point (*cf.* Art. 19, exercise 4).

8. If $ABCD$ is a harmonic range and a circle is described upon AC as diameter, of which S is any point, prove that the arc subtending the angle BSD, or its supplement, is bisected at A or at C.

28. Cross-ratio of a Harmonic Range.—If $ABCD$ is any range of points along a line, then the numerical value of the expression

$$\frac{AB}{BC} \div \frac{AD}{DC}$$

determined from the lengths and directions of the several segments involved, is called the **cross-ratio** or **anharmonic ratio** of the range.

Let $ABCD$ be a harmonic range. Project this range from any point S (Fig. 20) and cut the resulting pencil by a line through B parallel to SD. The resulting range $A_1BC_1\ \infty$ is harmonic (theorem IV). By similar triangles,

$$\frac{AB}{A_1B} = \frac{AD}{SD}$$

and

$$\frac{BC}{BC_1} = \frac{CD}{SD}.$$

Since $A_1B = BC_1$ (Art. 26), we obtain by division,

$$\frac{AB}{BC} = \frac{AD}{CD} = -\frac{AD}{DC}.$$

Hence: **The cross-ratio of a harmonic range is -1.**

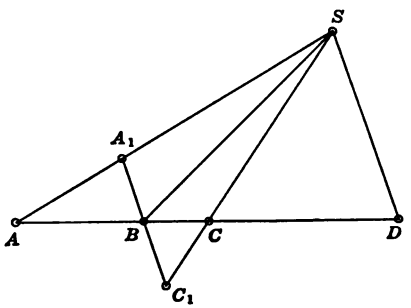


FIG. 20.

29. Harmonic Mean Between Two Numbers.—If three numbers are in arithmetic progression, their reciprocals are said to be in **harmonic progression.** For example, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$ are in harmonic progression, since 3, 5, 7 are three numbers of an arithmetic progression. In general, $1/a$, $1/b$, $1/c$ are in harmonic progression if a, b, c are three numbers of an arithmetic progression. Thus $1/b$ is the **harmonic mean** between $1/a$ and $1/c$ if b is the arithmetic mean (average) between a and c.

If $ABCD$ is a harmonic range of points, then (Art. 28)

$$\frac{AB}{BC} = \frac{AD}{CD} \text{ or } \frac{AB}{AC - AB} = \frac{AD}{AD - AC},$$

from which we easily find that

$$\frac{1}{AB} - \frac{1}{AC} = \frac{1}{AC} - \frac{1}{AD}.$$

Therefore the three fractions

$$1/AB,\ 1/AC,\ 1/AD$$

are in arithmetic progression and consequently the segments

$$AB,\ AC,\ AD$$

are in harmonic progression.

Hence; *If ABCD is a harmonic range, the segment AC is the harmonic mean between the segments AB and AD.*

Exercises

1. The points *ABC* are situated along a line at distances of one, three, and four units from *O*, respectively. How far from *O* is the harmonic conjugate of each point with respect to the other two?

2. Three points *ABC* are situated along a line so that *B* is two units from *A*, and *C* is three units from *A*. Find the position of a series of points, *DEFGH* . . ., so that *ABCD*, *ACDE*, *ADEF*, *AEFG*, . . . are harmonic ranges. The scale thus constructed is called a **harmonic scale.**

30. Geometric Mean Between Two Numbers.—If *a*, *b*, *c* are three numbers in geometric progression, then *b* is the **geometric mean** between *a* and *c*, or $b^2 = ac$.

Let *ABCD* be a harmonic range and *M* the mid-point of the segment *BD* (Fig. 21). Since

$$\frac{AB}{BC} = \frac{AD}{CD},$$

we have,

$$\frac{AM - BM}{BM - CM} = \frac{AM + BM}{BM + CM}.$$

Clearing of fractions and reducing, we get

$$\overline{BM}^2 = AM \cdot CM.$$

Hence: *If ABCD is a harmonic range and M is the mid-point of the segment BD, then BM is the geometric mean between AM and CM.*

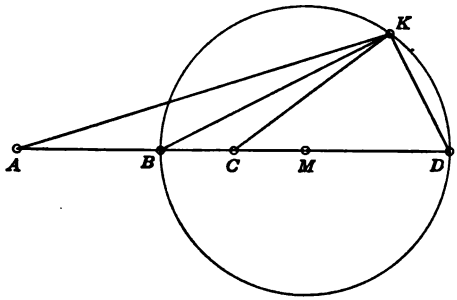


FIG. 21.

31. The Circle of Apollonius.—If we project the harmonic range $ABCD$ from a point K on the circle whose diameter is BD (Fig. 21), we obtain a harmonic pencil having two normal conjugate rays, KB and KD. Hence (Art. 27) KB bisects the angle AKC, and therefore

$$\frac{AK}{KC} = \frac{AB}{BC}.$$

The ratio $AK : KC$ is therefore constant so long as K is on the circle. In other words: *The locus of a point, the ratio of whose distances from two fixed points is a given constant, is a circle.* This circle is known as the **circle of Apollonius.**

32. Orthogonal Circles.—Let $ABCD$ be a harmonic range (Fig. 22). If through A and C we draw any circle and from M, the center of the circle of Apollonius, the two tangents MT and MT_1, we have from plane geometry,

$$\overline{MT}^2 = \overline{MT_1}^2 = AC \cdot CM.$$

But (Art. 30),

$$\overline{MB}^2 = AC \cdot CM.$$

Therefore,

$$MT = MT_1 = MB.$$

Consequently the points of contact of these tangents lie on the circle of Apollonius.

Again, if O is the center of the circle through A and C, the angle OTM = the angle $OT_1M = 90°$.

Therefore: *Any circle through A and C cuts the circle of Apollonius at right angles.*

The totality of circles that can be drawn through A and C forms a system of circles orthogonal to the circle of Apollonius.

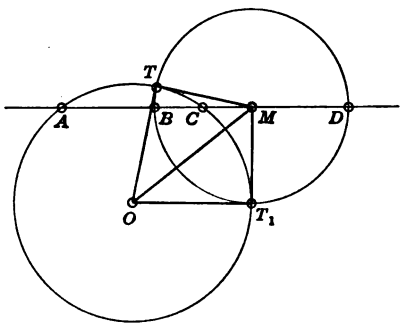


FIG. 22.

Exercises

1. Given two pairs of points, AB and CD, upon the same straight line which do not separate each other. With the aid of circles find two points which harmonically separate each pair.

Suggestion.—Choose any point, P, not on the line containing the given pairs. Draw the circles PAB and PCD intersecting again in the point E. Let the straight line PE meet AB in O. Draw a tangent from O to PAB or to PCD and let the point of contact be T. The circle whose center is O and whose radius is OT will cut the line AB in the required points (*cf.* Art. 32).

2. Given two pairs of points, AB and CD, upon the same straight line. Find the locus of points from which the segments AB and CD subtend equal angles.

Suggestion.—By the preceding exercise, find the two points, M and N, which separate harmonically each of the pairs AD and BC. From any point of the sphere whose diameter is MN, the segments AB and CD subtend equal angles (Art. 27).

3. If two pairs of points, AB and CD, upon the same straight line separate each other, find the locus of points from which AB and CD subtend equal angles.

4. If $ABCD$ is a harmonic range, show that

$$\frac{2}{AC} = \frac{1}{AB} + \frac{1}{AD}.$$

5. If *abcd* is a harmonic pencil of rays and (ab) indicates the angle which a makes with b, show that

$$\frac{\sin(ab)}{\sin(bc)} = -\frac{\sin(ad)}{\sin(dc)}.$$

6. If $ABCD$ is any range of points along a straight line, show that

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0.$$

7. If $ABCD$ is a harmonic range, show that the cross-ratio of the range $ACBD$ is 2; of $ADBC$ is $\frac{1}{2}$.

NOTES.—The idea of four harmonic points, or harmonic division, was known to the early Greek geometers, but who first invented it is not definitely known. Apollonius of Perga (247 B.C.) mentions it in his book on conic sections.

The harmonic property of a complete quadrangle is contained in the *Collections* of Pappus (300 A.D.). It was made the foundation for Von Staudt's *Geometrie der Lage*, 1847.

Three cords consisting of the same substance and having the same size and tension, and whose lengths are in harmonic progression, will vibrate in harmony when struck in unison. The name harmonic is probably due to this fact.

The theory of cross-ratios is due to Möbius (*Der Barycentrische Calcul*, 1827), to Steiner (*Systematische Entwickelung . . .*, 1832), and to Chasles (*Aperçu Historique . . .*, 1837). Möbius called the cross-ratio of four points the "double-ratio;" Chasles called it "anharmonic ratio." Some properties of the cross-ratio were known much earlier and are to be found in the *Collections* of Pappus.

CHAPTER V

PROJECTIVELY RELATED PRIMITIVE FORMS OF THE FIRST KIND

33. Primitive Forms of the First Kind.—The point-row, the sheaf of rays, and the sheaf of planes can be correlated, one with another, by the operations of projection and section (Art. 15). Moreover, no one of the remaining eight primitive forms can be derived by projection or section from any one of these three. Also the dual, or reciprocal, of any one of these three primitive forms is again one of the same three forms, and is never one of the remaining eight.

The point-row, the sheaf of rays, and the sheaf of planes thus constitute a group by themselves and are called **primitive forms of the first kind.** Primitive forms of the first kind are one dimensional forms (Art. 15).

34. Chains of Perspectivity.—A series of primitive forms so arranged that any one of them is in perspective position with the next in order constitutes a **chain of perspectivity.** Thus,

$$u \overline{\overline{\wedge}} S \overline{\overline{\wedge}} S_1 \overline{\wedge} u_1 \overline{\overline{\wedge}} S_2$$

is a chain of perspectivity.

The first and the last primitive form of any chain of perspectivity are not in general in perspective position with each other, but they are correlated, element to element, and by virtue of theorem IV, we can say that: *to any four harmonic elements of either there always correspond four harmonic elements of the other.*

Exercises

1. If $u \overline{\overline{\wedge}} S \overline{\overline{\wedge}} u_1$, are u and u_1 in perspective position?
2. Construct a figure illustrating the statement

$$u \overline{\overline{\wedge}} S \overline{\overline{\wedge}} S_1 \overline{\wedge} u_1.$$

If M is any point on u, construct the corresponding point on u_1.

3. A point P is joined to the vertices of a triangle ABC, the joining lines meeting the sides opposite A, B, C in D_1, E_1, F_1 respectively. The two triangles ABC and $D_1E_1F_1$ are in perspective position. Why? The axis of perspectivity meets the sides of the triangle ABC in the points D, E, F, the harmonic conjugates of D_1, E_1, F_1, with respect to the two vertices of ABC on the same side. Why? The axis meets the lines PA, PB, PC in the points R, S, T, respectively. Prove that the ranges $FRED$, $FSDE$, and $ETDF$ are harmonic. Also the ranges $RTSF$, $SRTD$, and $RSTE$ are harmonic.

35. Definition of Projective Relationship.—Two correlated primitive forms of the first kind are **projectively related** if to any four harmonic elements of either there correspond four harmonic elements of the other. Thus, if two primitive forms are in per-

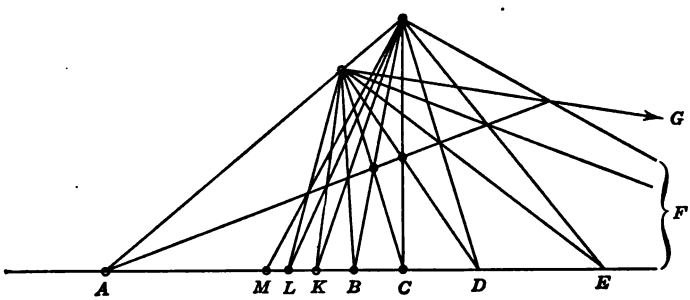


FIG. 23.

spective position, or if they belong to a chain of perspectivity, they are projectively related (theorem IV).

The symbol $\overline{\wedge}$ placed between two primitive forms indicates that they are projectively related. Thus,

$$S \overline{\wedge} S_1$$

means that the two sheaves of rays whose centers are S and S_1 are projectively related.

When two primitive forms are projectively related, we shall say that a **projectivity** exists between them. For example, a projectivity exists between the first primitive form and the last, in any chain of perspectivity.

36. Harmonic Scales.—With three points A, B, C of a point-row, chosen arbitrarily, we can determine a series of points D, E,

$F, G, \ldots, M, L, K$ (Fig. 23) such that $ABCD$, $ACDE$, $ADEF$, $\ldots$, $AMLK$, $ALKB$, $AKBC$ are harmonic ranges. The series of points thus constructed is a **harmonic scale.** Each point of the scale is harmonically separated from A by the two points which stand next to it. For example, L is harmonically separated from A by M and K. A is called the **origin** of the scale.

The projector of a harmonic scale on a point-row is a harmonic scale in a sheaf of rays; the projector of a harmonic scale in a sheaf of rays is a harmonic scale in a sheaf of planes. It follows, from theorem IV, that any section of a harmonic scale in a sheaf of rays (sheaf of planes) is a harmonic scale on a point-row (in a sheaf of rays).

If two point-rows are projectively related and we know three pairs of corresponding points, we can construct infinitely many pairs of corresponding points. For if A, B, and C correspond respectively to A_1, B_1, and C_1, the harmonic scales determined by ABC and $A_1B_1C_1$ must correspond, point to point, in the given projectivity (Art. 35).

Exercises

1. If A, B, and C are three points along a straight line, B being three units from A and C four units from A, construct the harmonic scale determined by A, B, and C.

2. Given any three points A, B, C of a point-row, construct the harmonic scale determined by them.

3. If the origin of a harmonic scale is the ideal point on the point-row, show that the points of the scale are equidistant from each other.

4. In exercise 1 find the cross-ratio of the range $BCDE$.

37. Two Pairs of Points Each Harmonically Separated by a Third Pair.—Theorem V.—*If a pair of points AC harmonically separates each of two other pairs of points, BD and B_1D_1, then the pair BD is not separated by the pair B_1D_1.*

Let $ABCD$ (Fig. 24) be a harmonic range and $KLMN$ a quadrangle defining it. Suppose the points A, K, L, and C remain fixed while M and N move along the fixed lines LC and KC respectively. The two variable points B and D will always be harmonically separated by A and C. To every position of B there is just one position of D (theorem III). When B moves in one direction along the line AC, D moves in the opposite direc-

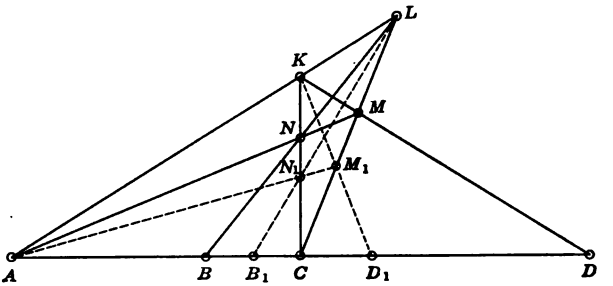


FIG. 24.

tion. As B describes the finite segment CA, D describes the infinite segment CA. Consequently, no two pairs, as BD and B_1D_1, can ever separate each other.

38. Converse of Theorem V.—*If two pairs of points, BD and B_1D_1, do not separate each other, then there always exists at least one pair of points AC which harmonically separates each of the others.*

Since B and D are not separated by B_1 and D_1, it follows that both B_1 and D_1 are on one of the segments formed by the pair B, D (Fig. 25).

Let A be a point on the segment BD not occupied by B_1 and D_1. Construct C_1, the harmonic conjugate of A with respect to B and

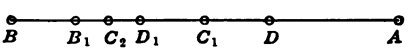


FIG. 25.

D, and also C_2, the harmonic conjugate of A with respect to B_1 and D_1. By theorem V, as A describes the segment BD upon which it lies, C_1 will describe the complementary segment BD and C_2 will remain upon the segment B_1D_1 not occupied by A. Consequently C_1 and C_2 must eventually coincide in at least one point C on the segment B_1D_1 not occupied by A, and then A and C harmonically separate both pairs, BD and B_1D_1 (*cf.* Art. 32, exercise 1).

Exercises

1. If A, B, C, D, are four points on a line such that A, B is not separated by C, D, nor A, D, by B, C, construct the pair of points harmonically separating A, B and C, D; also the pair harmonically separating A, D and B, C (*cf.* Art. 32, exercise 1).

2. By means of projection, show that theorem V and its converse apply to pairs of rays in a sheaf of rays, and to pairs of planes in a sheaf of planes.

39. Consequences of Theorem V and Its Converse.—With the aid of theorem V and its converse, we can reach some important conclusions concerning projectively related primitive forms of the first kind. For example, suppose u and u_1 are two projectively related point-rows, the points A, B, C, D of u corresponding respectively to the points A_1, B_1, C_1, D_1 of u_1. Then, if A, B does not separate C, D, it follows that A_1, B_1 cannot separate C_1, D_1. For we can find two points M and N which harmonically separate A, B and C, D (converse of theorem V). The ranges $AMBN$ and $MCND$ are then harmonic and, by the definition of projective relationship, correspond to harmonic ranges on u_1. Thus, if M_1, N_1 are the points corresponding to M, N, then $A_1M_1B_1N_1$ and $M_1C_1N_1D_1$ are harmonic ranges. Consequently A_1, B_1 cannot separate C_1, D_1 (theorem V).

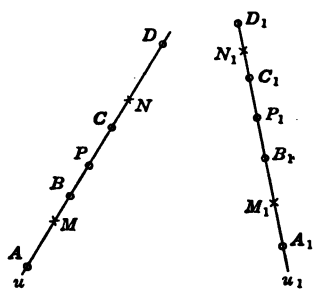


FIG. 26.

The property just proved for projectively related point-rows can be extended at once to any two projectively related primitive forms of the first kind by means of projection. In general, therefore; *If two primitive forms of the first kind are projectively related, any two non-separating pairs of elements of the one always correspond to two non-separating pairs of elements of the other.*

Returning to the case of two projectively related point-rows, we can say that any point P on the segment BC not occupied by A, D (Fig. 26), corresponds to a point P_1 on the segment B_1C_1 not occupied by A_1, D_1. For A, B and P, C, and also B, P and C, D,

are pairs of non-separating points, and hence A_1, B_1 cannot separate P_1, C_1, nor can C_1, D_1 separate B_1, P_1.

The operation of interpolating points on any segment of u can be continued indefinitely, and the corresponding points must occur on u_1 in exactly the same order as they are placed upon u. Thus the arrangement $ABPQ$——CD must correspond to the arrangement $A_1B_1P_1Q_1$——C_1D_1. This result is evidently immediately carried over to the other primitive forms of the first kind by means of projection.

If we commence with three points A, B, C upon u, we can construct by repeated use of harmonic ranges an indefinitely great number of points upon u. Theorem V and its converse enable us to say that the corresponding points upon the projectively related point-row u_1 must occur in the same order as they occur upon u.

But it has been shown[1] that construction by harmonic ranges, even if repeated an infinite number of times, can never construct all the points upon u. Theoretically, we may arrive at a point-row whose points are **everywhere dense**; that is, such that between any two of its points there are an infinite number of constructed points. But the point-row constructed in this way is not continuous. A familiar example from metric geometry may serve to illustrate. Suppose that B is one unit from A and we interpolate upon the segment AB all the points whose distances from A are proper fractions. The constructed points are everywhere dense, but the point whose distance from A is $\frac{1}{2}\sqrt{2}$ is on the segment and is not one of the constructed points.

From what has been said, it appears that theorem V and its converse enable us to say that, if $u \overline{\wedge} u_1$, a point-row upon u everywhere dense corresponds to a point-row upon u_1 also everywhere dense, the arrangement of points upon the two point-rows being the same.

To arrive at continuously projective point-rows use is made of the **Dedekind postulate,** which can be stated as follows:

If the infinite succession of points APQR——STB on any segment AB of u, constructed by repeated use of harmonic ranges, is divided into two sets such that:

[1] The proof is beyond the scope of this book. *Cf.* Clebsch-Lindemann, *Vorlesungen über Geometrie*, vol. ii, p. 433, where other references are given.

First,—any point of the succession A——B belongs to one of the two sets;

Second,—the terminal A belongs to the first set and B to the second;

Third,—any point whatever of the first set precedes every point of the second:

then there exists one, and but one, point M such that all points of the succession A——B that precede M belong to the first set, and all points of the succession A——B that succeed M belong to the second set.

The point M, whose existence is thus postulated, may belong to the first set in which case M is the last point of the set, or M may belong to the second set in which case it is the first point of the second set, or M may belong to neither set in which case the first set has no last point and the second set has no first point. In this last case M is a new point of the segment AB, not belonging to the succession A——B of constructed points, and is defined by the two sets of points into which the succession A——B is divided.

The division of the succession A——B into two sets of points effects a corresponding division in the succession $A_1P_1Q_1R_1$—— $S_1T_1B_1$ on the projectively related point-row u_1, and this division defines the point M_1 corresponding to M.

The Dedekind postulate thus enables us to add a series of new points on AB not constructible by means of harmonic ranges; and a repeated application of this postulate to the succession of points already constructed makes it theoretically possible to reach all the points on the segment AB and consequently all the points on the line u, since AB is any segment of u. To each point of u there is a corresponding point on u_1 arrived at by corresponding operations upon the constructed points of u_1.

Clearly, what has been said about the points of u and u_1 is immediately carried over by projection to sheaves of rays and sheaves of planes, and we are led, finally, to the important continuity theorem.

Theorem VI.—*If a moving element describes continuously either of two projectively related primitive forms of the first kind, the corresponding element must describe continuously the other.*

Exercise

In Fig. 27, $u \overline{\overline{\wedge}} S \overline{\wedge} u_1 \overline{\overline{\wedge}} S_1 \overline{\wedge} u_2$ and therefore $u \overline{\wedge} u_2$. Note that any point on the finite segment AB corresponds to a point on the infinite segment A_2B_2, and *vice versa*. If M is the ideal point on u, construct M_2; if N_2 is the ideal point on u_2, construct N.

40. Superposition.—Self-corresponding Elements.—Two point-rows are **superposed** when they lie upon the same straight line; two sheaves of rays are superposed when they are coplanar and concentric; two sheaves of planes are superposed when they have a common axis, or are coaxial.

Two projectively related primitive forms of the first kind, consisting of like elements, may be compared by superposition. To

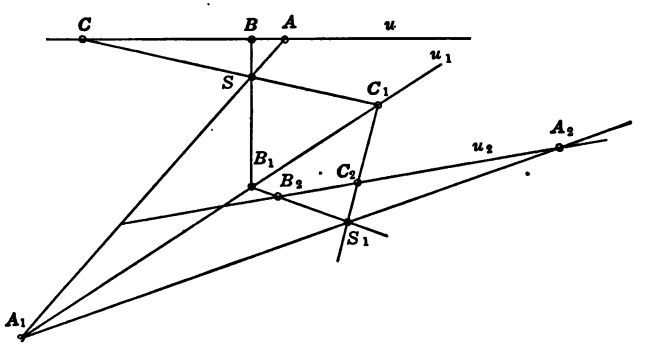


FIG. 27.

compare primitive forms in this way is the same kind of operation as that employed by the draughtsman when he compares his scale with a line to be measured, or his protractor with an angle to be measured.

When two projectively related primitive forms are superposed, it may happen that some elements of one of them coincide with their corresponding elements of the other. Such elements, if they exist, are called **self-corresponding** elements, or **double** elements. The possibility of the existence of at least one self-corresponding element is evident, for two projectively related primitive forms can be superposed in such a way that any particular element of

one falls upon its corresponding element of the other. This is actually done, for example, when two scales are compared by placing their zero points together. In certain cases, however, it is easy to see that there must be at least two self-corresponding elements. If, for example, u and u_1 (Fig. 28) are two projectively related point-rows, superposed so that the arrangement of points ABC—— runs from left to right while the corresponding arrange-

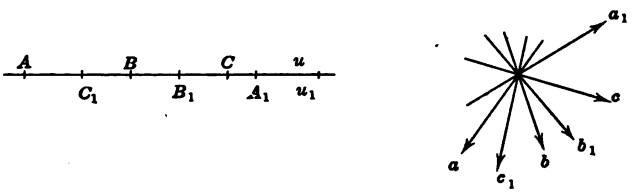


FIG. 28.

ment $A_1B_1C_1$—— runs from right to left, then it is evident that a variable point which describes one of them continuously must meet and pass its corresponding point at least twice. Thus, in this case, there must be at least two self-corresponding points. The point-rows themselves are said to be **oppositely projective.**

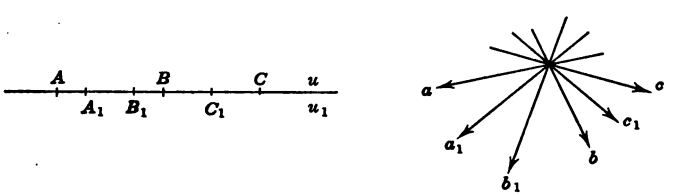


FIG. 29.

If the point-rows are **directly projective**; that is, if corresponding arrangements of points run in the same direction, there are not necessarily any self-corresponding points. If, however, any segment AB of u (Fig. 29) is included in the corresponding segment A_1B_1 of u_1 (or *vice versa*), it is easy to see that there must be at least one self-corresponding point.

Figs. 28 and 29 illustrate the two cases for projectively related and superposed sheaves of rays.

41. Von Staudt's Fundamental Theorem.—Theorem VII.—*If two projectively related primitive forms of the first kind are superposed and have three self-corresponding elements, then all their elements are self-corresponding and the two forms are consequently identical.*[1]

Consider at first two projectively related and superposed point-rows u and u_1, and let A, B, and C be the three self-corresponding points, so that A_1, B_1, C_1 fall respectively at A, B, C (Fig. 30). The harmonic scales determined by ABC and by $A_1B_1C_1$ must coincide, point for point, by virtue of the projective relationship. There is, then, an infinity of self-corresponding points. But this does not exclude the possibility that some point, not belonging to the harmonic scale determined by ABC, fails to coincide with its corresponding point. Suppose, for example, that a point P on the finite segment AB does not coincide with its corresponding point P_1. Since the arrangement $APBC$ must corre-

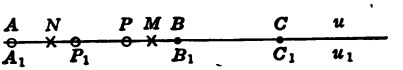


FIG. 30.

spond to the arrangement $A_1P_1B_1C_1$ (Art. 39), it follows that P_1 must lie on the same segment with P. As P moves continuously towards B, P_1 must move continuously toward B by virtue of the continuity theorem. P and P_1 must coincide, therefore, either at, or before, reaching B. Let us suppose they first coincide at M. Similarly, in moving toward A, let us suppose they first coincide at N. We shall then have a segment MN within the finite segment AB, or coinciding with it, on which no point coincides with its corresponding point. But this is impossible, since the harmonic conjugate of C with respect to M and N lies on the finite segment MN and coincides with its corresponding point by virtue of the projectivity between u and u_1. Consequently, every point is a self-corresponding point and the two point-rows are identical.

The same argument obviously holds for projectively related and superposed sheaves of rays or sheaves of planes. Or we may prove the theorem for these primitive forms by means of projection, having established it for superposed point-rows.

[1] Theorem VIII (Art. 43) is often called Von Staudt's fundamental theorem.

Theorem VII is of great importance in the further development of pure projective geometry. It is known as the fundamental theorem of Von Staudt.

42. Consequences of Theorem VII.—The principal results that depend upon theorem VII are stated in the following corollaries.

Corollary 1.—*If two distinct primitive forms of the first kind are projectively related and superposed, they cannot have more than two self-corresponding elements.*

For, if they have three self-corresponding elements, they are identical.

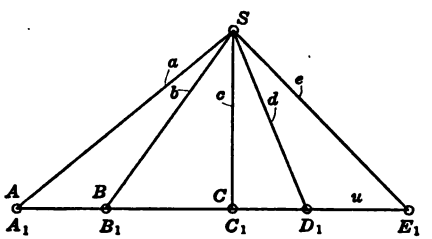


FIG. 31.

Corollary 2.—*If two primitive forms of the first kind consisting of unlike elements are projectively related and three elements of the first lie upon their corresponding elements of the second, then the two forms are in perspective position.*

Suppose $u(ABC\text{——}) \overline{\wedge} S(abc\text{——})$ (Fig. 31), the points A, B, C lying on their corresponding rays a, b, c. The line u cuts S in the point-row $u_1(A_1B_1C_1D_1\text{——})$. Hence, we have

$$u_1 \overline{\overline{\wedge}} S \overline{\wedge} u.$$

Therefore, to any harmonic range on u_1 there corresponds a harmonic range on u, and consequently u and u_1 are projectively related, superposed, and have three self-corresponding points; namely, A, B, and C. The point-rows u and u_1 are then identical and $S \overline{\overline{\wedge}} u$.

Exercise

If a sheaf of rays is projectively related to a sheaf of planes and three rays of the one lie in their corresponding planes of the other, prove that the two primitive forms are in perspective position.

Suggestion.—The plane of the sheaf of rays cuts the sheaf of planes in a second sheaf of rays. How are the two sheaves of rays related?

Corollary 3.—*If two projectively related point-rows lie in the same plane, but are not superposed, and have their common point as a self-corresponding point, then they are in perspective position.*

If two projectively related sheaves of rays lie in the same plane, but are not concentric, and have their common ray as a self-corresponding ray, then they are in perspective position.

For the corollary on the left, let $u(ABC\text{——}) \overline{\wedge} u_1(A_1B_1C_1\text{——})$, A coinciding with A_1 (Fig. 32). Let the lines BB_1 and CC_1

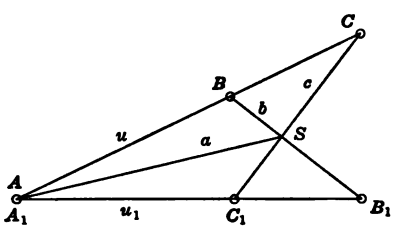


FIG. 32.

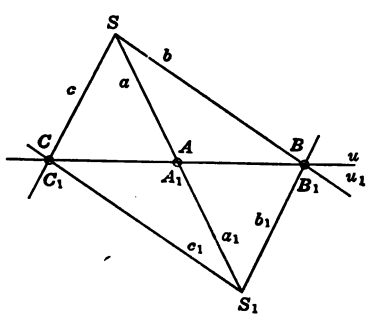


FIG. 33.

intersect in S. Project each point-row from S, thus obtaining two projectively related and superposed sheaves of rays having three self-corresponding rays, namely, a, b, and c. These sheaves are

consequently identical and the two point-rows are thus sections of the same sheaf of rays and therefore in perspective position.

Fig. 33 illustrates the dual corollary. The reader can easily supply the proof.

Corollary 4.—*If two projectively related sheaves of rays are not concentric and any three points of intersection of corresponding rays are collinear, then they are in perspective position.*

If two projectively related point-rows are not superposed and any three lines joining corresponding points are concurrent, then they are in perspective position.

For the corollary on the left, cut each sheaf by the line on which lie the three points of intersection. We thus obtain two projectively related and superposed point-rows having three self-corresponding points. The sheaves of rays are thus projections of the same point-row and consequently in perspective position.

The reader should supply the proof for the dual corollary.

Corollary 5.—*If two projectively related sheaves of planes whose axes are coplanar have their common plane as a self-corresponding plane, then they are in perspective position.*

Since the axes are coplanar, they meet in a point S. Cut the two sheaves by any plane not passing through S, and thus obtain two projectively related sheaves of rays which are coplanar and have their common ray as a self-corresponding ray (the trace of the self-corresponding plane). The sheaves of rays are, therefore, in perspective position (corollary 3), and thus projectors of the same point-row. Let $u(ABC\text{——})$ be this point-row. The two sheaves of planes are projectors of the sheaf of rays $S\,(ABC\text{——})$ and consequently they are in perspective position.

Exercises

1. Draw a figure illustrating corollary 5.

2. What is the space-dual of corollary 5?

3. Project Fig. 32 from a point not on the plane of the two point-rows. State and prove the corollary for the two projectively related sheaves of rays thus obtained. In the same way, project Fig. 33. What corollary is obtained from the resulting figure?

43. Determination of Projective Relationship.—Theorem VIII. —*A projectivity can always be established between any two primitive forms of the first kind so that any three elements of the one shall correspond to three elements of the other chosen arbitrarily. The projectivity thus established is unique.*

Suppose u and u_1 are two coplanar point-rows and we choose the points A, B, C of u to correspond respectively to A_1, B_1, C_1 of u_1 (Fig. 34). Join the points of any pair, for example B and B_1, and upon the line BB_1 choose any two points as S and S_1. From S project ABC——and from S_1 project $A_1B_1C_1$——.
Then if

$$u(ABC\text{——})\overline{\wedge}u_1(A_1B_1C_1\text{——}),$$

we have

$$S(abc\text{——})\overline{\overline{\wedge}}u(ABC\text{——})\overline{\wedge}u_1(A_1B_1C_1\text{——})\overline{\overline{\wedge}}S_1(a_1b_1c_1\text{——}).$$

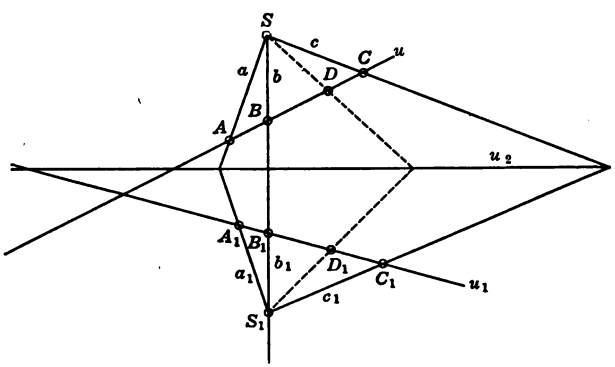


FIG. 34.

Hence S and S_1 are projectively related and have their common ray (b, b_1) as a self-corresponding ray. They are consequently in perspective position (corollary 3). The point-row of which both S and S_1 are projectors is constructed by joining the intersection of a and a_1 to the intersection of c and c_1. If u_2 is this point-row, then

$$u\overline{\overline{\wedge}}u_2\overline{\overline{\wedge}}u_1.$$

To any point as D of u we can now immediately construct the

corresponding point D_1 of u_1. For the projectors of D and D_1, from S and S_1 respectively, must intersect upon u_2.

That the point D_1 thus constructed is unique, is proved as follows. If by shifting the centers S and S_1 along BB_1, or if by choosing centers on AA_1 or on CC_1, we could arrive at a point D'_1, say, differing from D_1, we would have

$$A_1B_1C_1D_1 \text{——} \overline{\wedge} ABCD \text{——} \overline{\wedge} A_1B_1C_1D'_1 \text{——}.$$

and therefore

$$A_1B_1C_1D_1 \text{——} \overline{\wedge} A_1B_1C_1D'_1 \text{——}.$$

But this is impossible (theorem VII). Consequently D'_1 coincides with D_1; or, in other words, D_1 is uniquely determined.

Since D is any point on u, we can as above construct as many pairs of corresponding points as we choose. The projectivity is then completely determined.

Having proved the theorem for two coplanar point-rows, we may prove it for other pairs of primitive forms of the first kind by projection or section. For example, two coplanar sheaves of rays in which the rays abc are chosen to correspond to the rays $a_1b_1c_1$, each to each, may be cut by two lines on which the points ABC correspond to the points $A_1B_1C_1$, each to each. The projectivity established between these point-rows as above, establishes the projectivity between the sheaves of rays. Again, two sheaves of planes in which $\alpha\beta\gamma$ are chosen to correspond to $\alpha_1\beta_1\gamma_1$ can be cut by a plane in two coplanar sheaves of rays in which the traces of α; β, γ correspond respectively to the traces of α_1, β_1, γ_1. The projectivity established between these sheaves of rays establishes the required projectivity between the sheaves of planes.

On account of its importance, we shall prove the theorem for two coplanar sheaves of rays without reference to the proof just given for two coplanar point-rows. Thus, (Fig. 35), if

$$S(abc \text{——}) \overline{\wedge} S_1(a_1b_1c_1 \text{——}),$$

and we cut each sheaf by a line passing through the intersection of a pair of corresponding rays (a and a_1 in the figure) we obtain two point-rows $u(ABC\text{——})$ and $u_1(A_1B_1C_1\text{——})$ in perspective position (corollary 3). We can then immediately construct the

ray d_1 of S_1 corresponding to any ray d of S. For, if S_2 is the center of the sheaf, of which u and u_1 are sections, the points (du) and (d_1u_1) are collinear with S_2. That d_1 is uniquely determined by this process follows as in the case of two coplanar point-rows. If the process indicated could ever lead to a ray d'_1 not coinciding with d_1, then S_1 would be the common center of two projectively related sheaves of rays having the three self-corresponding rays $a_1b_1c_1$, but in which d_1 does not coincide with its corresponding ray d'_1, thus contradicting theorem VII.

Since d is any ray of S, the projectivity between S and S_1 is completely determined.

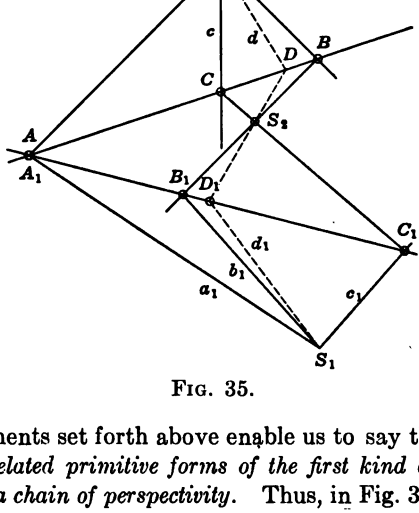


FIG. 35.

The arguments set forth above enable us to say that: *Any two projectively related primitive forms of the first kind can always be connected by a chain of perspectivity.* Thus, in Fig. 35,

$$S \overline{\wedge} S_2 \overline{\wedge} S_1.$$

Conversely; *if two primitive forms of the first kind are connected by a chain of perspectivity, they are projectively related* (*cf.* Art. 34).

Exercises

1. Given two coplanar point-rows u and u_1, determine the projectivity between them so that A, B, C of u shall correspond re-

spectively to A_1, B_1, C_1 of u_1: (*a*) when A, B, C and A_1, B_1, C_1, are all actual points; (*b*) when C and C_1 are ideal points. Construct a number of pairs of corresponding points in each case.

2. In the preceding exercise, construct the point on each point-row which corresponds to the point of intersection of the point-rows considered as a point on the other point-row. How many lines joining pairs of corresponding points can pass through any given point in the plane (corollary 4)?

3. Given two coplanar sheaves of rays S and S_1, determine the projectivity between them so that the rays a, b, c, of S shall correspond respectively to the rays a_1, b_1, c_1 of S_1. Construct a number of pairs of corresponding rays, in particular, the ray in each sheaf corresponding to the ray SS_1 considered as a ray of the other sheaf.

4. How can two projectively related sheaves of rays be placed in perspective position? Two projectively related point-rows?

5. If two sheaves of rays are coplanar and in perspective position, find two rays in one of them which are perpendicular to each other and which correspond to two perpendicular rays in the other. Show that in two projectively related sheaves of rays, whose centers are not ideal points, there is always a pair of homologous right angles.

Suggestion.—If $S\overline{\wedge}S_1$ and u is the axis of perspectivity, a circle can be drawn through S and S_1 whose center is on u.

6. Given two fixed straight lines u and u_1 intersecting in O, and two points S and S_1 collinear with O. A straight line rotates about a fixed point U and intersects u and u_1 in A and A_1 respectively. Show that the locus of the intersection of SA and S_1A_1 is a straight line passing through O (Chasles, *Geometrie Superièure*, 1880; also the *Collections* of Pappus). State the dual proposition.

7. If the three sides of a variable triangle BC, AC, AB rotate about three collinear points D, E, F, respectively, while two vertices A and B move upon two fixed straight lines which intersect in O, show that the third vertex C will describe a straight line passing through O (*cf.* Desargues theorem, Art. 17).

8. Prove the theorem stated in exercise 4, Art. 10.

Suggestion.—The sheaves A (DBF) and C (DBF) are in perspective position and cut out ranges on DE and FE, respectively, that are in perspective position with each other (corollary 3). L is the center of perspectivity for these ranges.

9. If the four vertices, A, B, C, D of a variable quadrangle move respectively upon four fixed lines which pass through one point O, while three of the sides AB, BC, CD, rotate about three fixed collinear

points, then the remaining three sides will also rotate about fixed points (Cremona, *Projective Geometry*, 1885).

NOTE.—The concept of projective relationship between two primitive forms is due to Möbius (*Der Barycentrische Calcul*, 1827).

In order to free projective geometry from any considerations of measurement, Von Staudt defined projective relationship as in Art. 34 (*Geometrie der Lage*, 1847), and based its further development upon the fundamental theorem (theorem VII) and its consequences.

CHAPTER VI

ELEMENTARY FORMS

44. Definition of Elementary Forms.—If two primitive forms of the first kind consisting of like elements are in perspective position, any pair of corresponding elements determines a third element. The locus of this third element is the primitive form of which the original two are either projectors or sections. Thus, if two sheaves of rays are in perspective position, the locus of the intersection of corresponding rays is the point-row of which the two sheaves of rays are projectors. Reciprocally, if two point rows are in perspective position, the locus of the rays joining corresponding points is the sheaf of rays of which the two point-rows are sections.[1]

Two primitive forms in perspective position are said to generate the primitive form of which they are either projectors or sections.

If two primitive forms of the first kind are projectively related but not in perspective position and any pair of corresponding elements determines a third element which varies with the pair, the locus of this third element is called an **elementary form.** The two primitive forms generate the elementary form. Thus, if two sheaves of rays are coplanar and projectively related, but not in perspective position nor concentric, any pair of corresponding rays determines a point. The locus of this point cannot be a straight line, for then the sheaves of rays would be projectors of this line and thus in perspective position. Reciprocally, if two point-rows are coplanar and projectively related, but not in perspective position nor superposed, the rays joining corresponding points cannot all pass through the same point, for then the point-rows would be in perspective position.

[1] If two non-intersecting point-rows are in perspective position, they are sections of infinitely many sheaves of planes. Two corresponding points do not determine any plane, and the point-rows do not generate any primitive form. This case will be considered in Chapter X.

We have, then, the following reciprocal definitions.

If two projectively related sheaves of rays are coplanar, but are neither concentric nor in perspective position, they generate a point-row of second order, or curve, which has not more than two points in common with any straight line.

If two projectively related point-rows are coplanar, but are neither superposed nor in perspective position, they generate a sheaf of rays of second class, or envelope, which has not more than two rays passing through any point.

For, if more than two points lie on any straight line (rays pass through any point), the two primitive forms are then in perspective position (corollary 4).

It may happen that a pair of corresponding elements does not determine a third element variable with the pair. The primitive forms do not then generate an elementary form. Thus, two projectively related sheaves of rays which are neither coplanar nor concentric do not generate an elementary form, for corresponding rays do not, in general, intersect. If, however, the sheaves of rays are concentric, corresponding rays determine a plane passing through the common center, and thus generate an elementary form. Again, projectively related sheaves of planes which are not in perspective position nor coaxial, always generate an elementary form since corresponding planes intersect in a line. Hence the following definitions:

Two projectively related sheaves of planes whose axes intersect, but which are not in perspective position, generate a cone of second order whose vertex is the common point of the axes. Not more than two rays of the cone lie in any plane passing through the vertex.

Two projectively related sheaves of lines which are concentric, but are neither coplanar nor in perspective position, generate a sheaf of planes of second class whose vertex is the common center. Not more than two planes of the sheaf intersect in any line passing through the vertex.

If three rays of the cone lie in any plane through the vertex, or three planes of the sheaf intersect in any line through the vertex, the generating primitive forms are in perspective position.

The two elementary forms just defined are reciprocals of each

other in the duality existing between a bundle of rays and a bundle of planes in which the ray and the plane are reciprocal elements.

Finally, we have the following definitions:

Two projectively related point-rows, which do not lie in the same plane, generate a regulus no two of whose rays can lie in any plane.

Two projectively related sheaves of planes, whose axes do not intersect, generate a regulus no two of whose rays can meet in any point.

The reciprocal of a regulus in space is again a regulus.

If, from each point-row as an axis, we project the other point-row, we obtain two projectively related sheaves of planes whose axes do not intersect. Corresponding planes in these two sheaves intersect in rays of the regulus. Reciprocally, if we cut each of two generating sheaves of planes by the axis of the other sheaf, we obtain two projectively related point-rows which generate exactly the same regulus.

If any two rays of a regulus lie in one plane, the point-rows which generate the regulus are coplanar; and the regulus itself becomes a sheaf of rays of the second class.

That the five elementary forms defined above are the only elementary forms generated by two projectively related primitive forms of the first kind, will appear in the sequel.

The words "second order" used in the above definitions signify that not more than two points (rays) can lie on a line (in a plane). Reciprocally, the words "second class" mean that not more than two rays (planes) can pass through a point (ray). From this point of view, the primitive forms of the first kind are of first order (class).

45. Fundamental Properties of the Curve and the Envelope.— Before discussing the relations existing between the five elementary forms, we shall develop some fundamental properties of the curve, or point-row of second order, and its reciprocal in the plane, the envelope, or sheaf of rays of the second class.

1. *A point-row of second order passes through the centers of the two projectively related sheaves of rays which generate it.*

A sheaf of rays of second class contains the supports of the two projectively related point-rows, which generate it.

For the line joining the centers of the sheaves, considered as a ray of one of the sheaves, must intersect its corresponding ray at the center of the other sheaf. Similarly, the point common to the two point-rows, considered as belonging to one of the point-rows, is joined to its corresponding point by the other point-row.

2. *The ray corresponding to the common line, considered as a ray of one of the sheaves, is tangent to the point-row of second order at the center of the other sheaf.*

. *The point corresponding to the common point, considered as a point of one of the point-rows, is the point of contact on the support of the other point-row.*

In the statement on the left, as a variable ray p describes the sheaf S (Fig. 36), its corresponding ray p_1 describes the sheaf S_1. The intersection P of p and p_1 describes the curve passing through S and S_1 (property 1). As P approaches S_1, p approaches the position n (or SS_1) and the secant S_1P (or p_1) approaches a limiting position n_1, corresponding to n, which is defined as the **tangent** at S_1. Again, as P approaches S, p_1 approaches the position of the common ray S_1S, and p approaches a limiting position m, defined as the **tangent** at S.

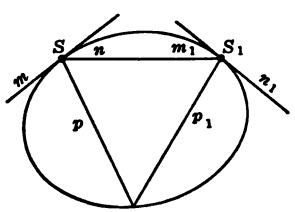


FIG. 36.

In the statement on the right, as a variable point P describes the point-row u (Fig. 37), its corresponding point P_1 describes the point-row u_1. The line joining P and P_1 describes the sheaf of rays of second class, or envelope, which contains the rays u and u_1. As P approaches the common point M of u and u_1, P_1 approaches the point M_1 corresponding to M. M_1 is called the **point of contact** on u_1. Similarly, as P_1 approaches the common point of u and u_1 (which we now call N_1 as belonging to the point-row u_1), P approaches the corresponding point N on u. N **is the point of contact** on u.

A point-row of second order has a tangent at each one of its points and a sheaf of rays of second class has a point of contact on each one of its rays. Methods for constructing the tangent at

any point of a curve and the point of contact on any ray of an envelope will be developed later.

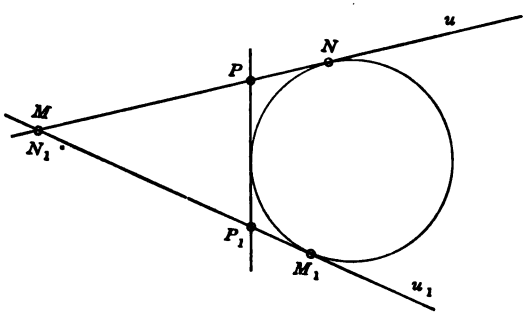


FIG. 37.

3. *A point-row of second order is projected from any two of its points in projectively related sheaves of rays in which corresponding rays intersect on the point-row.*

A sheaf of rays of second class is cut by any two of its rays in projectively related point-rows in which corresponding points are joined by rays of the sheaf.

Suppose the point-row of second order is generated by the two projectively related sheaves of rays S and S_1 (Fig. 38). Choose any four points on the point-row, as A, B, C, D, then

$$S(ABCD) \barwedge S_1(ABCD)$$

by definition. Cut the sheaves S and S_1 by the lines DC and DA respectively, and thus obtain two projectively related point-rows along these lines in which the ranges $DCTR$ and $DMK\mathcal{A}$ correspond point to point. These point-rows are in perspective position, since their common point D corresponds to itself (corollary 3). Therefore the lines MC, KT, and RA are concurrent in S_2. If the point D is allowed to describe the curve while the points S, S_1, A, B, C remain fixed, the lines DC and DA will describe sheaves of

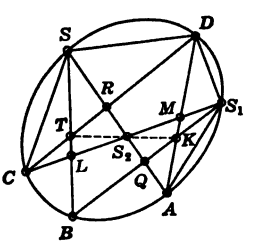


FIG. 38.

rays about C and A respectively. At the same time, the points T and K will describe point-rows along the fixed lines SB and S_1B respectively. But these point-rows are in perspective position, since they are sections of the sheaf whose center is the fixed point S_2. Hence, we have the chain of perspectivity

$$C(BS_1DS)\overline{\overline{\wedge}}BLTS\overline{\overline{\wedge}}BS_1KQ\overline{\overline{\wedge}}A(BS_1DS).$$

Consequently

$$C(BS_1DS)\overline{\wedge}A(BS_1DS)$$

and corresponding rays meet in points of the curve. Thus the curve is also generated by the sheaves whose centers are C and A. But C and A are any two points of the curve.

For the reciprocal statement, let the sheaf of rays of second class be generated by the two projectively related point-rows u and u_1 in which $ABCD$ and $A_1B_1C_1D_1$ are corresponding ranges, so that AA_1, BB_1, CC_1, and DD_1 are any four rays of the sheaf of second class (Fig. 39). Let AA_1 intersect BB_1 and DD_1 in the points S_1 and S respectively. Project u and u_1 from S and S_1 respectively, and thus obtain two sheaves of rays in perspective position, since their common ray corresponds to itself. These sheaves are then projectors of the point-row u_2. If the ray AA_1 is allowed to describe the sheaf of second class, the points BCD and $B_1C_1D_1$ remaining fixed, the points S_1 and S will describe point-rows along the fixed lines BB_1 and DD_1 in perspective position with the sheaves of rays described by S_1C_1 and SC respectively. Hence,

$$D_1SDL\text{——}\overline{\overline{\wedge}}C(D_1RBT\text{——})\overline{\overline{\wedge}}C_1(D_1RBT\text{——})\overline{\overline{\wedge}}B_1S_1BM\text{——}.$$

Consequently

$$D_1SDL\text{——}\overline{\wedge}B_1S_1BM\text{——}$$

and corresponding points are joined by rays of the sheaf of second class. But BB_1 and DD_1 are any two rays of this sheaf.

As an immediate consequence of the property stated in (3), we have the following:

4. *A point-row of second order is completely determined by any five of its points.*

A sheaf of rays of second class is completely determined by any five of its rays.

For, if $ABCDE$ are any five points of a point-row of second order, we can choose any two of them as centers of projectively related sheaves of rays and these will then generate the curve. If A and B are the centers chosen, then the sheaves are $A(CDE)$

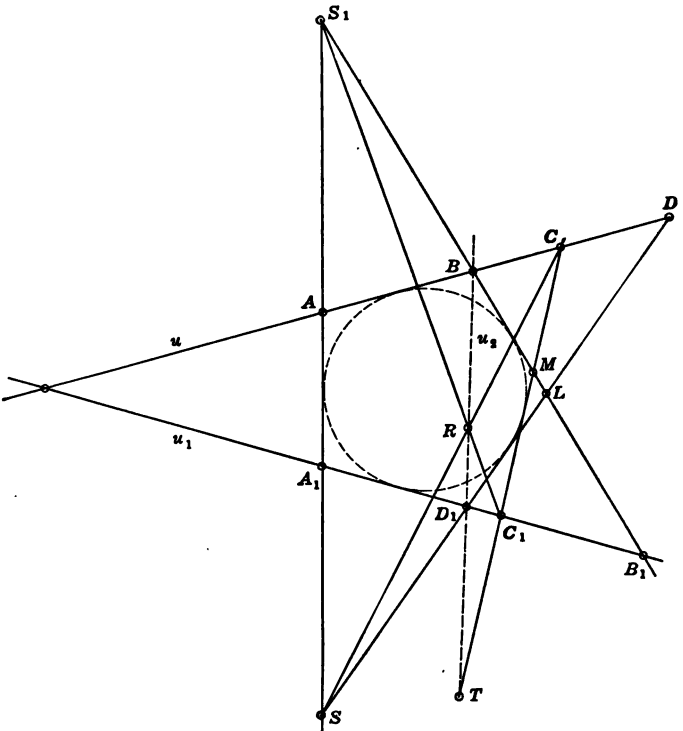


FIG. 39.

and $B(CDE)$. The three pairs of corresponding rays determine the projectivity (Art. 43, theorem VIII).

Reciprocally, if $a\ b\ c\ d\ e$ are five rays of a sheaf of rays of second class, a projectivity is determined between any two of them by the remaining three. As many rays of the sheaf of second class can then be constructed as may be desired.

46. Construction of Curves and Envelopes.—By means of the properties stated in Art. 45 and with the aid of theorem VIII, we are able to solve the following construction problems, viz.:

To construct a curve of second order having given (a) five points, (b) four points and the tangent at one of them, (c) three points and the tangents at two of them.

To construct an envelope of second class having given (a) five rays, (b) four rays and the point of contact on one of them, (c) three rays and the points of contact on two of them.

For, in each case, the given elements are just sufficient to determine a projectivity between two primitive forms and these will

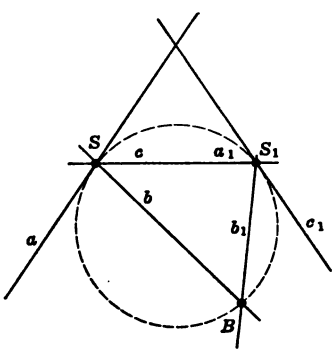


FIG. 40.

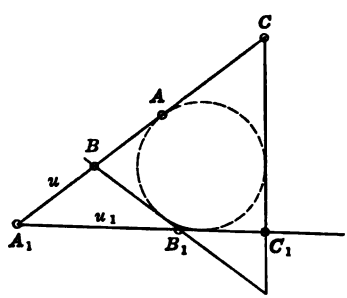


FIG. 41.

then generate the elementary form required. Thus, in case (*c*) on the left, let S, S_1 and B be the three given points and a and c_1, the tangents at S and S_1 respectively (Fig. 40). The three pairs of corresponding rays, aa_1, bb_1, cc_1, completely determine a projectivity between the sheaves of rays S and S_1 (theorem VIII), and these generate a curve of second order which passes through S, S_1, and B and is tangent to a and c_1 at S and S_1 respectively, by virtue of the properties stated in (1) and (2), Art. 45. As many points of the curve can be constructed as may be desired.

Again, in case (*b*) on the right, if AC, A_1C_1, BB_1, and CC_1 are the four given rays and A is the given point of contact on AC (Fig. 41), then the pairs of points A, A_1, B, B_1, and C, C_1 determine

a projectivity between the rays AC and A_1C_1. These projectively related point-rows generate a sheaf of rays of second class containing the four given rays and having A as the point of contact on AC. As many rays of this sheaf can then be constructed as may be desired.

Special cases may arise. Thus, if three of the five given points in (*a*) on the left are collinear, the sheaves formed by projecting any three of the points from the remaining two are in perspective position and thus generate a point-row of the first order (Art.)44.

Reciprocally, if three of five given rays are concurrent, the point-rows cut by any three of the rays upon the remaining two are in perspective position and thus generate a sheaf of rays of the first class.

The construction in each case is unique (theorem VIII) and hence we conclude that:

Two curves of second order coincide if they have in common five points, or four points and the tangent at one of them, or three points and the tangents at two of them.

Two envelopes of second class coincide if they have in common five rays, or four rays and the point of contact on one of them, or three rays and the points of contact on two of them.

Exercises

1. Construct the point-row of second order determined by five given points two of which are ideal.

2. Construct the sheaf of rays of second class determined by five given rays one of which is the ideal line.

3. Construct the point-row of second order determined by three points, two of which are ideal points, and the tangents at the ideal points.

4. The centers of two sheaves of rays lie on a circle and corresponding rays meet in points of the circle. Can the sheaves be superposed so as to coincide throughout? Are the sheaves projectively related? Is the circle a curve of second order?

5. A variable tangent to a circle cuts out point-rows on two fixed tangents to the same circle. What angle does the segment of the variable tangent contained between the fixed tangents subtend at the center of the circle? Is this angle constant? Are the point-rows cut out on the fixed tangents projectively related? Is the system of tangents to a circle a sheaf of rays of second class?

6. A variable triangle ASA_1 moves in its plane so that the vertices A and A_1 continually lie upon straight lines u and u_1 respectively, while the sides SA and SA_1 rotate about the fixed point S, the angle S remaining of constant magnitude. Show that the base AA_1 describes a sheaf of rays of second class to which u and u_1 belong.

7. Given a plane quadrangle $KLMN$. What is the locus of a point S such that the pencil $S(KLMN)$ is always harmonic?

8. If a, b, c, d, e, are five rays of an envelope of second class, construct the points of contact on b and c.

9. If A, B, C, D, E are five points of a curve of second order, construct the tangents at A and B.

10. In the preceding exercise, A and B are ideal points. The tangents at A and B are *asymptotes*. Construct the asymptotes.

11. If the point of contact on any ray of an envelope of second class is infinitely distant, that ray is an *asymptote*. Construct an envelope of second class having given two asymptotes and one other ray.

12. A triangle moves so that its vertices continually lie upon three fixed lines while two of its sides rotate about two fixed points. Show that the third side will, in general, describe an envelope of second class. Under what circumstances will the envelope be of first class?

13. What is the reciprocal of the preceding exercise in the plane?

14. Two angles P and Q of fixed magnitude and lying in the same plane rotate about P and Q while the intersection of two sides traverses a fixed line. Show that the intersection of the other two sides will, in general, describe a curve of second order (Newton's method of generating conics).

47. Relations Existing among the Elementary Forms.—The elementary forms are related one to another by means of projection and section. Thus, the section of a cone, by a plane not passing through its vertex, is a curve of second order. For the plane cuts the two sheaves of planes which generate the cone in two sheaves of rays which generate the curve. Corresponding planes are cut in corresponding rays, the latter intersect upon that ray of the cone which is determined by the former.

Similarly, the section of a sheaf of planes of second class, by a plane not passing through its vertex, is an envelope of second class. For the plane cuts the two sheaves of rays which generate the sheaf of planes in two point-rows which generate the envelope. Corresponding rays are cut in corresponding points, and the latter are joined by the trace of the plane determined by the former.

Again, if we project a curve of second order from a point not lying in its plane, we obtain a cone of second order. For the projectors of the two sheaves of rays which generate the curve are two sheaves of planes which generate the cone.

In the same way, the projector of an envelope of second class is a sheaf of planes of second class, since the projectors of the point-rows that generate the envelope are sheaves of rays that generate the sheaf of planes.

For the regulus, we have the following dual statements:

The section of a regulus, by a plane not passing through any ray, is a curve of second order. For the plane cuts the generating sheaves of planes in sheaves of rays which generate the curve.	*The projector of a regulus, from a point not lying on any ray, is a sheaf of planes of second class. For the projectors of the generating point-rows are sheaves of rays which generate the sheaf of planes.*

That the elementary forms are also related one to another by means of the principle of duality has been exemplified in the definitions (Art. 44) and in the properties developed in Art. 45. There are, however, other reciprocal relations of importance. Thus, the reciprocal of a curve of second order in space is a sheaf of planes of second class, since:

The curve is composed of points lying in a plane, no three of which can lie on any straight line.	*The sheaf is composed of planes passing through a point, no three of which can meet in any straight line.*

Again, an envelope of second class is the space-dual of a cone of second order, since:

The envelope consists of rays lying in a plane, no three of which can meet in any point.	*The cone consists of rays passing through a point, no three of which can lie in any plane.*

Exercises

1. By means of projection, derive the properties of a cone and of a sheaf of planes of second class corresponding to the properties of a curve and of an envelope stated in Art. 45.

2. Derive the same properties by means of the principle of duality.

48. Totality of Elementary Forms.—That the elementary forms defined in Art. 44 are the only ones generated by two projectively related primitive forms of the first kind follows upon comparing the primitive forms of the first kind, one with another, in all possible ways. We have already considered the cases where the two primitive forms consist of like elements. The only other cases in which two projectively related primitive forms of the first kind can generate an elementary form are:

1. A sheaf of rays projectively related to a sheaf of planes.
2. A sheaf of rays projectively related to a point-row.

In the first case, if α is the plane containing the given sheaf of rays, the elementary form is likewise generated by the given sheaf of rays and the section of the sheaf of planes by α, and is, therefore, a curve of second order.

In the second case, if S is the center of the given sheaf of rays, the elementary form is also generated by the given sheaf of rays and the projector of the point-row from S, and is consequently a sheaf of planes of second class.

We conclude, therefore, that there are only five elementary forms generated by projectively related primitive forms of the first kind, namely, those defined in Art. 44.

49. Classification of Curves of Second Order.—Since a curve of second order cannot meet any straight line in more than two points (Art. 44), it cannot have more than two points in common with the ideal line in its plane. If the curve does not meet the ideal line, it is called an **ellipse**: if it has one point in common with the ideal line; that is, if it is tangent to the ideal line, it is called a **parabola**: and if it meets the ideal line in two distinct points, it is called a **hyperbola.**

The ellipse lies wholly in the finite part of the plane, the parabola stretches out indefinitely in one direction toward the infinitely distant part of the plane, while the hyperbola crosses the ideal line and appears on both sides of it.

The tangents to the hyperbola at the ideal points are called the **asymptotes.** If the asymptotes are perpendicular to each other, the hyperbola is **equilateral.**

50. The Conic Sections.—That the ellipse, the parabola, and the hyperbola are sections of a cone of second order whose vertex

is not an ideal point may be seen as follows: A plane α passing through the vertex must meet the cone in this point alone, or touch the cone along one of its rays s, or cut the cone in two of its rays p and q. Any plane parallel to α must, then, cut all the rays of the cone in actual points, or cut all the rays except s in actual points, or cut all the rays except p and q in actual points. In the first case, the section of the cone is an ellipse; in the second, a parabola; and in the third, a hyperbola. Hence, in the future, we can call the curves of second order **conic sections,** or more briefly **conics.**

A further development of the properties of the conics and their reciprocals, the envelopes of second class, requires the aid of two important and historic theorems which are reciprocals of each other. These theorems will be the subject of the next chapter.

Exercises

1. If the axes of two projectively related sheaves of planes are parallel to each other, what elementary form is generated by them?

2. If from a given point perpendicular lines are dropped upon the planes of a sheaf of first class, what is the locus of the feet of these perpendiculars?

3. If from a given point perpendicular lines are dropped upon the planes of a sheaf of second class, show that these lines lie upon a cone of second order.

4. If from a point outside the plane of a sheaf of rays of first class, perpendicular planes are drawn to the rays of the sheaf, these planes will form a sheaf of planes of first class. What line is the axis of the sheaf?

5. Construct a hyperbola having given the two asymptotes and one point.

Suggestion.—Take the ideal points on the given asymptotes as the centers of two projectively related sheaves of rays.

6. If two projectively related sheaves of rays are coplanar, but not concentric, and are oppositely projective (Art. 40), show that they must generate a hyperbola. If they are directly projective, would they necessarily generate an ellipse? Why?

CHAPTER VII

THE PASCAL THEOREM AND THE BRIANCHON THEOREM

51. Six Elements of a Conic or of an Envelope.—Five points in a plane, no three of which lie on any line, completely determine a conic (Art. 45). Reciprocally, five rays in a plane, no three of which pass through any point, completely determine an envelope. New properties of the conics and of their reciprocals, the envelopes, are derived from the geometrical condition that must be satisfied in order that a sixth element may belong to the conic, or to the envelope, determined by any given five elements. The condition for the conic is known as the **Pascal theorem,** and the corresponding condition for the envelope is called the **Brianchon theorem,** from their respective discoverers.

Theorems IX.—The Pascal Theorem.—*The opposite sides of any simple hexagon whose vertices are points of a curve of second order intersect in three collinear points.*

The Brianchon Theorem.—*The opposite vertices of any simple hexagon whose sides are rays of an envelope of second class are joined by three concurrent lines.*

For the Pascal theorem, let A, B, C, D, E, F be the vertices of the hexagon (Fig. 42). Since the conic can be generated by sheaves of rays whose centers are any two of these points (Art. 45), we may choose A and C as the centers. The sheaves are then A ($BDEF$——) and C($BDEF$——). Cut these sheaves by the lines ED and EF respectively, and thus obtain the point-rows $LDEM$—— and $SREF$——. These point-rows are in perspective position (Art. 42, corollary 3), and hence LS, DR, and MF meet in a point U. Thus the opposite sides of the hexagon $ABCDEF$ intersect on the line LUS.

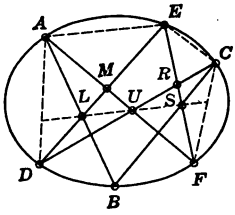


FIG. 42.

For the Brianchon theorem, let a, b, c, d, e, f be the sides of the hexagon (Fig. 43). Since the envelope of second class is cut by any two of these lines in projectively related point-rows (Art. 45), we may cut the envelope by the lines a and c, and thus obtain the projectively related point-rows $ABCD$—— and $A_1B_1C_1D_1$——. Let the sides d and f meet the side e in the points S and S_1 respectively. Project $ABCD$—— from S and $A_1B_1C_1D_1$—— from S_1, and we obtain two sheaves of rays in perspective position (corollary 3) whose corresponding rays meet on the line B_1D. But SA and

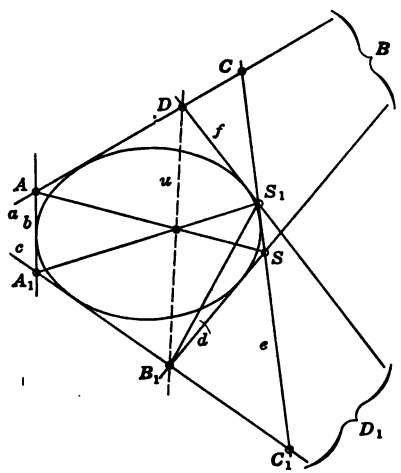


FIG. 43.

S_1A_1 are corresponding rays. Therefore opposite vertices of the hexagon are joined by concurrent lines.

52. Converse Theorems.—The statements of the Pascal theorem and of the Brianchon theorem in the preceding article are the ordinary ones. The proofs show that the conditions embodied in the statements are **necessary** in order that six elements shall belong to a curve or to an envelope. To show that these conditions are also **sufficient,** we must prove the converse of each theorem. That is, for the Pascal theorem, we must show that:

If the opposite sides of any simple hexagon intersect in three collinear points, the vertices are points of a curve of second order.

There is no difficulty in doing this. Thus, if A, B, C, D, E, F are the vertices of any simple hexagon (Fig. 42), the sheaves $A(BDF——)$ and $C(BDF——)$ generate a conic passing through the five points A, B, C, D, F (Art. 45). If the opposite sides of the hexagon meet in three collinear points, the point-rows cut upon the lines ED and EF by the sheaves $A(BDF——)$ and $C(BDF——)$, respectively, are in perspective position, since they are sections of the sheaf whose center is the point U in which the lines AF and CD intersect. Consequently the point E corresponds to itself in the two point-rows, and the rays AE and CE correspond to each other in the sheaves which generate the curve. Hence the conic passes through E, and thus contains all six of the vertices of the hexagon.

A special case arises if the vertices B, D, F are collinear. In this case, the sheaves $A(BDF——)$ and $C(BDF——)$ are in perspective position (corollary 4) and thus generate a point-row of first order passing through the points B, D, F. But since the rays AE and CE correspond to each other, as above, they are the projectors of the same point on the point-row $BDF——$, and hence the vertices A, C, E are also collinear (*cf.* Art. 10, exercise 4, also Art. 43, exercise 8). In this case, the curve of second order is said to **degenerate** into two point-rows of first order, namely, the point-rows $DBF——$ and $ACE——$. The figure consists of nine points and nine lines so situated that three points lie on each line and three lines pass through each point. This arrangement of points and lines in a plane is known as a **configuration of Pappus** (*cf.* configuration of Desargues, Art. 13).

The converse of the Brianchon theorem can be proved in a like manner. There is also a special case if three sides of the hexagon meet in a point. The resulting combination of points and lines is a configuration of Pappus. In this special case the envelope of second class is said to **degenerate** into two sheaves of rays of first class.

Exercise

State and prove the converse of the Brianchon theorem. Also state and prove the special case when three sides of the hexagon meet in a point. Draw figures to illustrate.

53. Application of Theorems IX.—The Pascal theorem and the Brianchon theorem enable us to solve the following construction problems:

Given five points of a curve of second order; with the aid of a ruler to determine the second point of the curve upon any straight line drawn through one of the given points.

Given five rays of an envelope of second class; with the aid of a ruler to determine the second ray of the envelope passing through any point on one of the given rays.

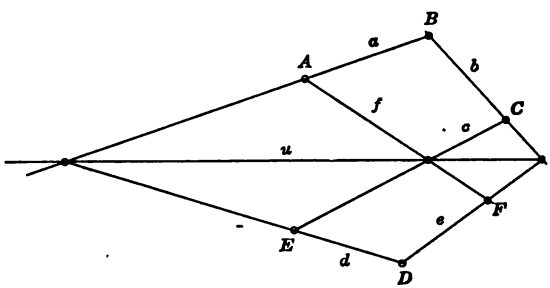


FIG. 44.

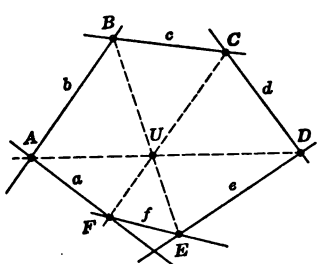


FIG. 45.

Let A, B, C, D, E be the five given points and suppose any line, as f, is drawn through one of them, say A (Fig. 44). Join these points two and two so as to form with f five sides of a

Let a, b, c, d, e be the five given rays and suppose any point, as F, is chosen on one of them, say a (Fig. 45). Find the intersection of these lines two and two so as to form with

simple hexagon. Let a, b, c, d be the lines so constructed. The problem now is to construct the sixth side, e, so that opposite sides of the hexagon intersect in three collinear points. If we arrange the sides thus:

$$a\ b\ c$$
$$d\ e\ f,$$

then the opposite sides are readily seen to be a and d, b and e, c and f. But the points (ad) and (cf) are joined by the line u. Hence (be) lies on u, and (ef), or F, is the point of the curve sought.

The line u is called a *Pascal line.*

By drawing a series of lines through A, we can construct as many points of the curve as may be desired.

F five vertices of a simple hexagon. Let A, B, C, D be the points so determined. The problem now is to construct the sixth vertex, E, so that opposite vertices of the hexagon are joined by three concurrent lines. If we arrange the vertices thus:

$$A\ B\ C$$
$$D\ E\ F,$$

then the opposite vertices are readily seen to be A and D, B and E, C and F. But the lines AD and CF intersect in the point U. Hence BE passes through U, and EF, or f, is the ray of the envelope sought.

The point U is called a *Brianchon point.*

By choosing a series of points on a, we can construct as many rays of the envelope as may be desired.

Exercises

1. A conic is given by five points in a plane of which two are ideal points; construct the second point of intersection with the conic of any line through one of the ideal points.

2. A conic is given by five points of which one is an ideal point; construct the second point of intersection of the conic with the ideal line.

3. A sheaf of rays of second class is given by five rays of which one is the ideal line in the plane; determine the ray of the sheaf which passes through any given ideal point. By choosing a series of ideal points, construct a number of rays of the sheaf.

54. Degenerate Cases of Theorems IX.—If, in the Pascal theorem, two vertices of the hexagon approach coincidence, the side joining them approaches a limiting position which is the

tangent at that point of the conic through which it passes. The hexagon then degenerates into a pentagon with the tangent at one vertex.

It is clear that the hexagon may also degenerate into a quadrangle with the tangents at two of the vertices, or into a triangle with the tangents at the three vertices.

In the Brianchon theorem, two sides of the hexagon may approach coincidence; their point of intersection then approaches a limiting position which is the **point of contact** on that ray of the envelope which passes through it. The hexagon then degenerates into a pentagon with the point of contact on one of its sides.

The hexagon may also degenerate into a quadrangle with the points of contact on two of its sides, or into a triangle with the points of contact on the three sides.

These degenerate cases of theorems IX and some applications of them are taken up in the following articles.

55. The Pentagon Theorem and Its Dual.—

If the vertices of any simple pentagon are points of a given curve of second order, then two pairs of non-adjacent sides intersect in points collinear with the intersection of the remaining side and the tangent at the opposite vertex.

For, if a, b, c, d, e are the sides of the pentagon (Fig. 46) and f is the tangent at one vertex, say (bc), then the sides of the degenerate hexagon are in order a, b, f, c, d, e. The three points (ac), (bd), (ef) are collinear by the Pascal theorem; a and c, b and d are pairs of non-adjacent sides and the vertex (bc) is opposite the side e.

If the sides of any simple pentagon are rays of a given envelope of second class, then two pairs of non-adjacent vertices are joined by lines concurrent with the line joining the remaining vertex to the point of contact on the opposite side.

For, if A, B, C, D, E are the vertices of the pentagon (Fig. 47) and F is the point of contact on one side, say BC, then the vertices of the degenerate hexagon are in order A, B, F, C, D, E. The three lines AC, BD, EF are concurrent by the Brianchon theorem; A and C, B and D are pairs of non-adjacent vertices and the side BC is opposite the vertex E.

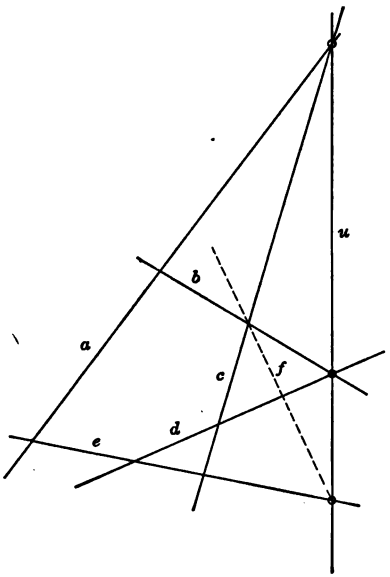


FIG. 46.

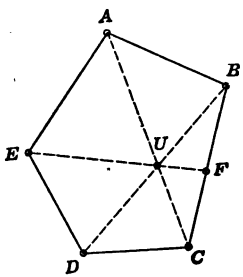


FIG. 47.

56. Application of the Pentagon Theorem and Its Dual.—

A curve of second order is given by five points; with the aid of a ruler, construct the tangent to the curve at each of the given points.

The five given points are vertices of a simple pentagon. Consider in turn each degenerate hexagon whose sides are the five sides of the pentagon and the tangent at one vertex.

An envelope of second class is given by five rays; with the aid of a ruler, construct the point of contact on each of the given rays.

The five given rays are sides of a simple pentagon. Consider in turn each degenerate hexagon whose vertices are the five vertices of the pentagon and the point of contact on one side.

Exercises

1. Given five points of a curve of second order of which two are ideal points; construct the tangents at the ideal points.

2. Given five rays of an envelope of second class of which one is the ideal line; construct the points of contact on each of the given rays.

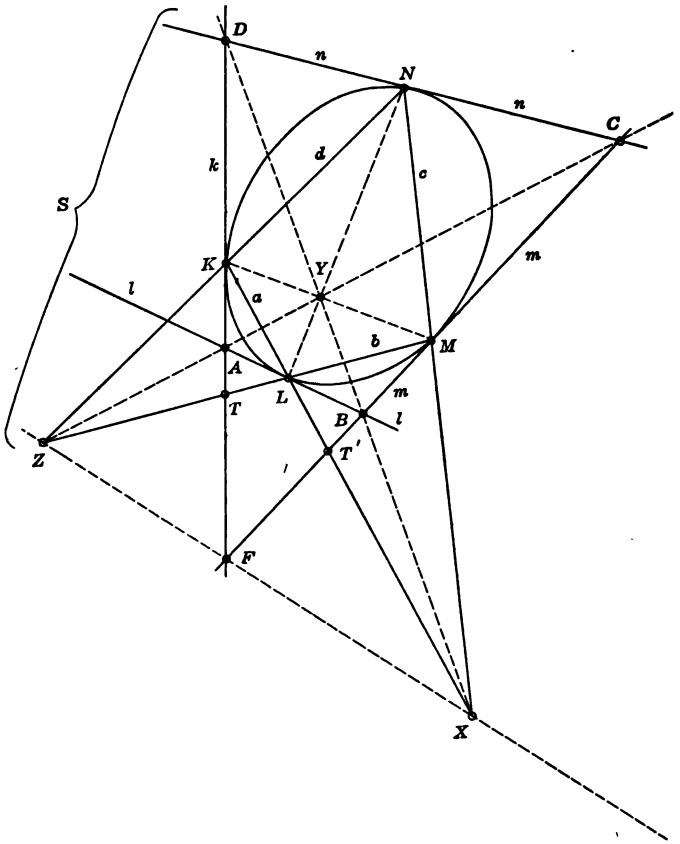


FIG. 48.

3. Given three points of a conic and the tangents at two of them. Through one of the points a straight line is drawn; construct the second point of intersection of this line with the curve.

4. Given four rays of an envelope of second class and the point of contact on one of them. A point is chosen on one of the given rays;

construct the second ray of the envelope which passes through this point.

57. The Quadrangle Theorem and Its Dual.—

If the vertices of any simple quadrangle are points of a given curve of second order, then pairs of opposite sides of the quadrangle intersect in points collinear with the points of intersection of tangents at opposite vertices.

The sides of the quadrangle together with the tangents at a pair of opposite verticles form a degenerate hexagon. Thus (Fig. 48), *akdcmb* and *adncbl* are degenerate hexagons of of which XZ is the common Pascal line.

If the sides of any simple quadrangle are rays of a given envelope of second class, then the diagonals of the quadrangle are concurrent with the lines joining the points of contact on opposite sides.

The vertices of the quadrangle together with the points of contact on a pair of opposite sides form a degenerate hexagon. Thus (Fig. 48), $AKDCMB$ and $ADNCBL$ are degenerate hexagons of which Y is the common Brianchon point.

58. Application of the Quadrangle Theorem and Its Dual.—

The tangents to a curve of second order form a sheaf of rays of second class.

The points of contact on the rays of an envelope of second class form a curve of second order.

If, in Fig. 48, we allow the point K to move along the curve while the points L, M, and N remain fixed, the lines DB and AC will describe sheaves of rays about the fixed points B and C respectively. But these sheaves are in perspective position, since corresponding rays intersect on the fixed line LN, and they cut out projectively related point-rows along the fixed tangents l and n respectively. Corresponding points on these point-rows are joined by the various positions of the tangent k. Thus k describes an envelope of second class.

On the other hand, if we allow k to describe an envelope of second class while l, m, and n remain fixed, the sheaves described about the fixed points B and C by the varying lines DB and AC will cut out point-rows along the fixed lines LM and MN respec-

tively. But these point-rows are in perspective position, since corresponding points, as X and Z, are joined by rays through the fixed point S. Project the point-rows thus determined on LM and MN from the fixed points N and L, respectively, and we obtain projectively related sheaves whose corresponding rays intersect in the various positions of the point of contact K. Therefore, K describes a curve of second order.

The theorems just proved show the relation between a curve of second order and an envelope of second class. From it we see that not more than two tangents to a conic can pass through any point in the plane.

A point through which pass two tangents to a conic is said to be **outside** the curve. If no tangents pass through the point, it is **inside** the curve. If one tangent passes through the point, it is **on** the curve and is the point of contact of the tangent which passes through it.

59. The Principle of Continuity.—The proofs given for the special cases in the preceding articles depend upon the so-called **principle of continuity.** This principle was formulated by Poncelet (1822) and asserts that properties of a geometrical figure which hold when the figure varies according to definite laws will also hold when the figure assumes a limiting position. Thus, the Pascal theorem holds as long as the six vertices of the hexagon are on the curve, and the principle of continuity asserts that the theorem holds also when the hexagon assumes a limiting position; that is, one or another of the degenerate cases stated in Art. 54. But the principle of continuity rests upon intuitive grounds rather than upon logical rigor. It is not necessary, however, to enter upon a discussion of its validity, since any one of the special cases is easily proved quite independently of the Pascal, or the Brianchon, theorem.

On account of its importance, we shall demonstrate the quadrangle theorem and its dual without reference to the Pascal theorem or the Brianchon theorem.

60. Second Proof of the Quadrangle Theorem and Its Dual.—

Let $KLMN$ be any four points of a curve of second	Let $klmn$ be any four rays of an envelope of second class

order and k and m the tangents at K and M respectively (Fig. 48). The curve is then generated by the two projectively related sheaves of rays $K(FNML)$ and $M(KNFL)$ (Art. 45). When we cut these sheaves by the lines ML and KL, respectively, we obtain the two point-rows $TZML$ and $KXT'L$ which are in perspective position, since they have the point L as a self-corresponding point. The point F is the center of perspectivity. Hence, the intersection of the tangents at the opposite vertices K, M lies on the line joining the intersections of pairs of opposite sides, viz., the line XZ. Similarly, with L and N as centers, we can show that the tangents at L and N also intersect on XZ.

and K and M the points of contact on k and m respectively (Fig. 48). The envelope is then generated by the two projectively related point-rows $AKDF$ and $BFCM$ (Art. 45). When we project these point-rows from B and A, respectively, we obtain the two sheaves $B(AKDF)$ and $A(BFCM)$ which are in perspective position, since they have the line AB as a self-corresponding ray. The line KM is the axis of perspectivity. Hence, the line joining the points of contact on the opposite sides k, m passes through the intersection of the lines joining opposite vertices, viz., the point Y. Similarly, with l and n as point-rows, we can show that the line joining the points of contact on l and n also passes through Y.

61. Generation of Particular Conics and Envelopes.—The conic generated by two projectively related sheaves of rays will be an ellipse, a parabola, or a hyperbola according as the sheaves have no pair, one pair, or two pairs of corresponding parallel rays. If one of the sheaves is superposed upon the other in such a way that the direction of its rays is unchanged, then these sheaves will have, in the first case, no self-corresponding rays; in the second case, one; and in the third case, two such rays. If the sheaves are oppositely projective (Art. 40), they will always generate a hyperbola.

The envelope generated by two projectively related point-rows will be the system of tangents to a parabola if the ideal points on

the point-rows correspond to each other, for then the ideal line is one of the tangents to the curve. If the two point-rows are placed in perspective position, by superposing any two actual homologous points (Fig. 49), they will be sections of a sheaf of parallel rays. Corresponding segments along the two point-rows are thus seen to be proportional to each other. Hence, *any two tangents to a parabola are cut proportionally by the remaining tangents.* For this reason, projectively related point-rows whose ideal points correspond to each other are called **similarly projective.**

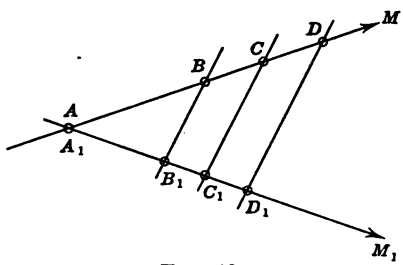


FIG. 49.

62. Cones and Sheaves of Planes of Second Class.—Since the tangents to a curve of second order form a sheaf of rays of second class, it follows by projection that the tangent planes to a cone form a sheaf of planes of second class.

Corresponding to the theorems of Pascal and Brianchon, there are correlative theorems for the cone and the sheaf of planes of second class which are at once derived by projecting the figures for the Pascal and Brianchon theorems from a point outside the plane in which they lie.

63. Cylinders.—A cone whose vertex is an ideal point is a **cylinder.** Two sheaves of planes which are projectively related, but not in perspective position, and whose axes are parallel generate a cylinder.

A cylinder is elliptic, parabolic, or hyperbolic according as a plane, not passing through its ideal vertex, cuts it in an ellipse, a parabola, or an hyperbola.

Exercises

1. If a point-row u and a sheaf of rays S are coplanar and projectively related, and through each point of u is drawn a straight line parallel to the corresponding ray of S; show that these lines will either intersect in one point or will envelope a parabola.

2. Construct a hyperbola of which there are given the asymptotes and one point or one tangent.

3. Construct a parabola of which there are given four tangents, or three tangents and the point of contact on one of them, or two tangents and their points of contact.

4. Given a point-row u and a sheaf of rays S which are coplanar and projectively related; show that straight lines drawn from the points of u perpendicular to the corresponding rays of S will either envelope a parabola or pass through one point.

5. If the vertices of a triangle move upon three fixed lines of the plane in such a manner that two sides of the triangle do not alter their directions; show that the third side will either envelope a parabola or move parallel to itself.

6. If a triangle is inscribed in a conic, show by the Pascal theorem that the tangents at the vertices meet the opposite sides in three collinear points (triangle theorem).

7. If a triangle is circumscribed about a conic, show that the lines joining each vertex to the point of contact on the opposite side are concurrent (dual of the triangle theorem).

8. Prove the pentagon theorem without making use of the Pascal theorem.

9. If a hexagon whose vertices are not coplanar nor its three diagonals concurrent is projected from any point on a line which meets all three of the diagonals, show that the lines projecting the vertices are rays of a cone of second order.

NOTE.—The Pascal theorem was discovered by Pascal when he was only 16 years of age. It was first published in a little work entitled *Essai pour les coniques* in which it was called the theorem of the mystic hexagram (1640). Pascal proved his theorem first for the circle and then, by means of projection, extended it to any conic.

Brianchon discovered the dual theorem in 1806 and published it in the *Journal de l'Ecole polytechnique*. This was before the principle of duality had been enunciated by Poncelet and Gergonne.

CHAPTER VIII

POLES AND POLAR LINES WITH RESPECT TO A CURVE OF SECOND ORDER

64. Poles and Polar Lines.—The theory of poles and polar lines with respect to a fixed conic is based upon the quadrangle theorem and its dual (Art. 57). The following dual theorems are fundamental.

Theorems X.—

If X is a point in the plane of a fixed conic, the harmonic conjugate of X, with respect to the curve points on any secant drawn through X, lies on a fixed straight line called the **polar line** *of X with respect to the conic.*

If x is a line in the plane of a fixed conic, the harmonic conjugate of x, with respect to the tangents to the conic from any point of x, passes through a fixed point called the **pole** *of x with respect to the conic.*

Thus, in Fig. 50, LM and KN are any two secants drawn through X. Pairs of opposite sides of the inscribed quadrangle $KLMN$ meet in the points X, Y, Z. The harmonic range $MVLX$ is defined by the quadrangle $ZKYN$, and the harmonic range $NUKX$, by the quadrangle $ZLYM$. Hence, the harmonic conjugates of X with respect to the curve points on the secants KN and LM lie on the line ZY. We now have to show that this line is fixed in position irrespective of the secants drawn through X. To do this, we recall that the tangents at opposite vertices of the simple quadrangle $KMNL$ intersect on YZ; that is, the points A and B lie on YZ (Art. 57). Hence, if we fix the secant LM and thus fix the points A and V, the line YZ is fixed irrespective of the position of

the secant XKN. The harmonic conjugate, U, must then lie on the fixed line YZ, whatever position is taken by the secant XKN. Or, if we fix the secant KN, the fixed points B and U determine

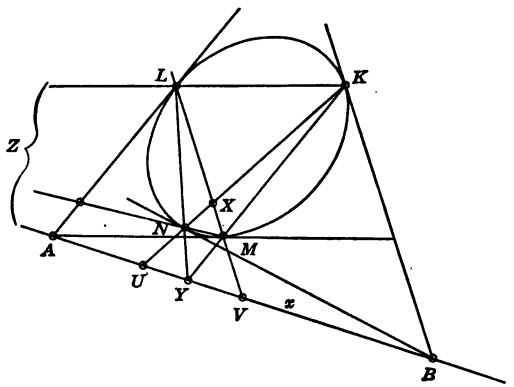


FIG. 50*a*.

the position of the line YZ; and V, the harmonic conjugate of X with respect to L and M, lies on YZ whatever position is taken by the secant XLM. Therefore the position of the line YZ is independent of the secants drawn through X.

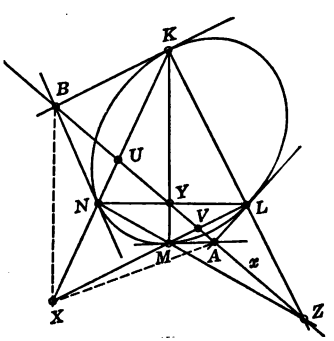


FIG. 50*b*.

For the dual theorem, A and B are any two points on x from which tangents can be drawn to the conic. Let K, L, M, N be the points of contact of these tangents; then (Art. 57) the lines KM and LN intersect on the diagonal x of the simple circumscribed quadrangle whose sides are the tangents at K, L, M, N. Also, the lines KL and NM intersect on x, since x is a diagonal of the quadrangle formed by the tangents at K, N, L, M. The lines KN and LM intersect in a point X through which pass the

harmonic conjugates of x with respect to the tangents from A and from B, since the pencils $B(NUKX)$ and $A(MVLX)$ are harmonic. Hence, if B is fixed, X is determined by the fixed line NK and the harmonic conjugate of x with respect to the tangents from B and if A, is fixed, X is determined by the fixed line ML and the harmonic conjugate of x with respect to the tangents from A. Consequently, the position of X is independent of the positions of A and B upon x.

65. Special Positions of Pole and Polar Line.—Since a point X is separated harmonically from points of its polar line by the conic, it follows that, if X is inside the curve (Fig. 50 (a)), its polar line lies wholly outside the curve; and if X is outside the conic (Fig. 50 (b)), its polar line cuts the curve.

Again, if X lies on the conic, all the harmonic conjugates of X with respect to curve points coincide with X (*cf.* Art. 37). The polar line of X is then the tangent to the conic at X.

Similarly, if a line x does not meet the fixed conic, its pole lies inside the curve. For then the harmonic conjugates of x, with respect to tangents drawn from points of x, all meet the curve. But if x meets the conic, none of these harmonic conjugates meets the curve, and the pole is consequently outside the conic.

If x touches the conic, the harmonic conjugate of x with respect to the tangents from any point of x coincides with x, since x is one of the two tangents. The pole is then the point of contact of x with the conic.

In conclusion, therefore, we can say:

With respect to a fixed conic, every point in the plane has a definite polar line; and every line in the plane has a definite pole.

66. Chords of Contact.—If X is a point outside a conic, the line joining the points of contact of tangents from X is called the **chord of contact** of these tangents.

If in Fig. 50 (b), the secant KN is allowed to rotate about X until the curve points coincide, the secant then becomes a tangent to the conic from X. But the curve points can only coincide on the polar line of X, since $XKUN$ is constantly a harmonic range. Similarly, the curve points L and M can only coincide on the polar line. Hence:

If X is a point outside a conic, its polar line with respect to the conic is the chord of contact of the tangents drawn from X.

If x cuts a conic, its pole with respect to the conic is the intersection of the tangents at the points in which x meets the conic.

67. Construction of Poles and Polar Lines.—If a conic is fully drawn, the fundamental theorems in Art. 64 will serve to determine the polar line of a given point with respect to the conic, or the pole of a given line with respect to the conic. Thus (Fig. 50), with X as the given point, the pairs of points U and V, or Y and Z, or A and B can be constructed. Any one of these pairs of points determines the polar line of X.

With x as the given line, the pairs of lines AX and BX, or a pair of diagonals of a circumscribed quadrilateral, one of whose diagonals is x, or the lines LM and KN can be constructed. Any one of these pairs of lines determines the pole of x.

If the given point is outside the conic (the given line cuts the conic), Art. 66 may be used to construct the polar line (the pole).

If a conic is not fully drawn, but is given by any one of the sets of conditions in Art. 46, the polar line of a given point with respect to the conic (or the pole of a given line) can be constructed with the aid of a ruler. Thus, if only five points of a conic are known, the polar line of any given point can be constructed as follows: Join the given point to any two of the known points, and, with the aid of the Pascal theorem, construct the points where these lines meet the conic again (Art. 53). We now have an inscribed quadrangle two of whose sides pass through the given point and we can complete the construction by Art. 64.

If only five tangents to a conic are known, we can construct the points of contact on these by Art. 56 and then proceed as before.

If only five tangents to a conic are known, we can construct the pole of any given line by finding the points of intersection of this line with two of the tangents and then (Art. 53). constructing a complete circumscribed quadrilateral one of whose diagonals is the given line. The other two diagonals intersect in the required pole.

If the conic is given by five points, we can construct the tangents at these (Art. 56) and proceed as before.

If any particular case, the general procedure outlined above is to be followed. For example, suppose it is required to construct the polar line of a given point when the conic is given by three tangents and the points of contact on two of them. Let a, b, and

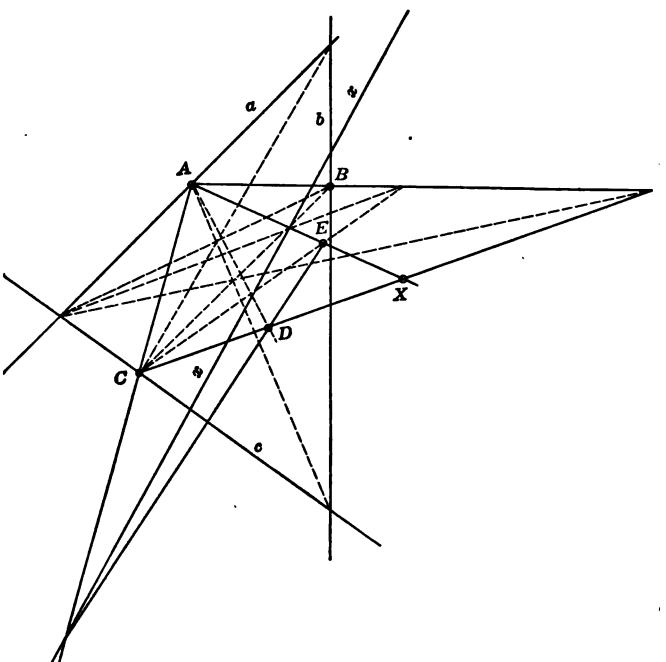


FIG. 51.

c be the three tangents and A and B the points of contact on a and b respectively (Fig. 51). Construct the point of contact, C, on c (Art. 63, exercise 7). Let X be the given point whose polar line is required. Join X to two of the points of contact, say A and C, and determine the points where the lines XA and XC meet the conic

again; that is, construct the inscribed quadrangles $ABCD$ and $AECB$. In the first quadrangle, the Pascal line is determined by the intersection of the lines AB, XC and the intersection of the tangents at A and C. The sides BC and AD meet on this line. Hence, D is determined. Similarly, E is determined. We now have an inscribed quadrangle two of whose sides meet in the given point X, viz., the quadrangle $AEDC$. The sides AD, CE and the sides AC, DE intersect on the required polar line.

Exercises

1. Construct the polar line of a given point with respect to a conic which is given by five tangents.

2. A conic is fully drawn, construct the tangents to it from an exterior point.

3. A conic touches four given lines one of which is the ideal line. The point of contact on the ideal line being given, construct the polar line of a given point; the pole of a given line.

4. A conic is given by five points two of which are ideal, construct the pole of a given line; the polar line of a given point.

5. A conic is given by three points and the tangents at two of them, construct the pole of a given line.

68. Conjugate Points and Conjugate Lines with Respect to a Conic.—If, in the plane of a fixed conic, two points are so situated that each lies on the polar line of the other, the points are said to be **conjugate points** with respect to the conic; and if two lines are so situated that each passes through the pole of the other, the lines are called **conjugate lines** with respect to the conic.

The following dual theorems are fundamental for conjugate points and conjugate lines.

Theorems XI.—

If one point lies on the polar line of another, the two points are conjugate with respect to the conic.	*If one line passes through the pole of another, the two lines are conjugate with respect to the conic.*

For the theorem on the left, if Y is any point of the polar line of

X, we are to prove that the polar line of Y passes through X. There are three cases to consider.

1. If X is inside the conic (Fig. 50), it is separated harmonically from every point of its polar line (and therefore from Y) by the curve. But Y is separated harmonically from all points of its polar line that are inside the conic (and from no other points inside the conic) by the curve. Since Y is separated harmonically from X by the curve, and X is inside the conic, X is on the polar line of Y.

2. If X is on the conic, Y is on the tangent to the conic at X (Art. 65). The polar line of Y is the chord of contact of tangents to the conic from Y and consequently passes through X.

3. If X is outside the conic, Y is on the chord of contact of tangents from X (Art. 66). But the tangents at the curve points on any secant through Y intersect on the polar line of Y (Art. 67). Therefore, X is on the polar line of Y.

Hence, in every case, the polar line of Y passes through X; and X and Y are, therefore, by definition, conjugate points with respect to the conic.

For the theorem on the right, if y passes through the pole of x, the pole of y lies on x by what has just been proved. Therefore, x and y are conjugate lines with respect to the conic.

69. Consequences of Theorems XI.—A given point is conjugate to all the points on its polar line and a given line is conjugate to all the lines through its pole. Hence, if a point describes a given line, its polar line will rotate about the pole of the given line. Likewise, if a line rotates about a given point, its pole will describe the polar line of the given point.

Thus, a given point-row has a sheaf of polar lines; and a given sheaf of rays has a point-row of poles.

1. *A point-row x is projectively related to its sheaf of polar lines X.*

Thus, in Fig. 50, let X, L, and M be fixed while N describes the curve. The points Z and Y will then describe projectively related point-rows along the fixed line x, since the rays MN and LN describe projectively related sheaves of rays about M and L respectively (Art. 45, 3). But the point-row described by Y is projected from X by the sheaf of rays described by XY which is the polar line of Z. Hence, the point-row described by Z is

projectively related to the sheaf of rays described by the polar line of Z.

2. *If U and V are two non-conjugate points in the plane, for every ray p of U there is a ray p_1 of V conjugate to it. The two sheaves of rays thus constructed are projectively related and generate a curve of second order, or a point-row of first order, according as the line UV does not, or does, touch the fixed conic.*

If u and v are two non-conjugate lines in the plane, for every point P of u there is a point P_1 of v conjugate to it. The two point-rows thus constructed are projectively related and generate an envelope of second class, or a sheaf of rays of first class, according as the point (uv) does not, or does, lie on the fixed conic.

For, on the left, each sheaf is the projector of the point-row of poles belonging to the other; and on the right, each point-row is a section of the sheaf of polar lines belonging to the other.

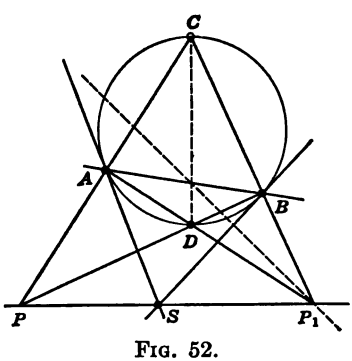


FIG. 52.

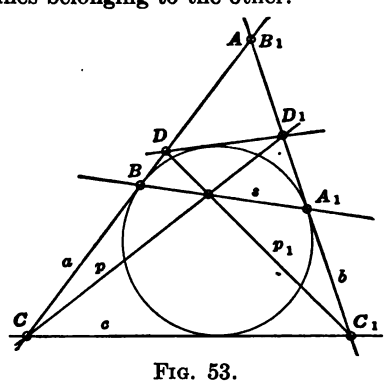


FIG. 53.

3. *If a triangle is inscribed in a conic, any line conjugate to one side with respect to the conic cuts the other two sides in conjugate points; and conversely.*

For, if ABC is any triangle inscribed in a conic and S is the pole of AB (Fig. 52), then the

If a triangle is circumscribed about a conic, any point conjugate to one vertex with respect to the conic is projected from the other two vertices by conjugate lines; and conversely.

For, if abc is any triangle circumscribed about a conic and s is the polar line of (ab) (Fig. 53),

sheaves of rays $A(CBS—)$ and $B(CSA—)$ are projectively related and cut the sides BC and AC, respectively, in point-rows which are in perspective position, S being the center of perspectivity. Any line through S, therefore, cuts these point-rows in a pair of corresponding points, as P and P_1. The lines PB and P_1A intersect on the conic and form sides of an inscribed quadrangle $ABCD$. Hence, the polar line of P passes through P_1.

then the point-rows $ABC—$ and $A_1B_1C_1—$ are projectively related and are projected from the vertices C_1 and C, respectively, in sheaves of rays which are in perspective position, s being the axis of perspectivity. Any point on s, therefore, is projected from C and C_1 by a pair of corresponding rays, as p and p_1. The points D and D_1 are joined by a tangent to the conic and form vertices of a circumscribed quadrilateral DCC_1D_1. Hence, the pole of p lies on p_1.

70. Polar Figures with Respect to a Fixed Conic.—By virtue of Art. 65 and with the aid of theorems XI, we can say that, with respect to a fixed conic, any plane figure consisting of points and lines has a **polar figure** consisting of lines and points. Thus, a triangle ABC has a polar triangle whose sides are the polar lines of A, B, and C and whose vertices are the poles of the sides of ABC.

Again (Art. 69, 1), we see that the polar of a harmonic range of points is a harmonic sheaf of rays.

If we regard a plane curve as described by a moving point, the polar line of the point will envelope the polar curve. Thus, if a point describes a curve of second order, its polar line will envelope a curve of second class. For the point is the intersection of corresponding rays in two projectively related sheaves of rays and hence its polar line joins corresponding points in two projectively related point-rows.

A fixed conic may thus be thought of as **polarizing** the plane in which it lies in such a way that to every line corresponds a definite point and to every point corresponds a definite line; and, further, if a point lies on a line, the corresponding line passes through the corresponding point.

As another example, consider a simple hexagon inscribed in a conic (Fig. 54). The polar figure with respect to the conic is a simple circumscribed hexagon, since the polar line of a point on the conic is the tangent at that point; and the pole of any chord is the intersection of the tangents at the extremities of the chord.

A pair of opposite sides of the inscribed hexagon intersect in a point whose polar line joins the corresponding opposite vertices of the circumscribed hexagon. Since pairs of opposite sides of the inscribed hexagon intersect in collinear points (Pascal theorem), it follows that pairs of opposite vertices of the circumscribed hexagon are joined by concurrent lines (Brianchon theorem). Thus either the Pascal theorem or the Brianchon theorem can be derived

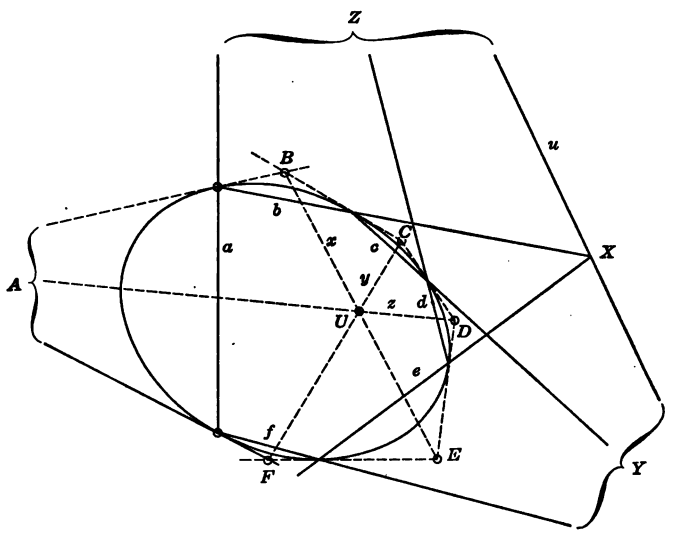


FIG. 54.

one from the other, by means of polar figures with respect to a fixed conic. The Pascal line thus appears as the polar line of the Brianchon point.

The concept of polar figures with respect to a fixed conic is but a special case of the concept of dual, or reciprocal, figures in the plane. Polar figures may thus be thought of as dual, or reciprocal, figures with respect to a fixed conic. With respect to a fixed conic, the dual, or reciprocal, of a given figure is perfectly definite, and can be constructed, while the general principle of duality, as

we know, merely asserts the existence of a dual figure without stating laws for its construction.

71. Self-polar Figures.—If a plane figure coincides with its polar figure, it is called **self-polar.** Thus two polar triangles, considered as a single figure, form a self-polar figure.

Again (Fig. 55), the triangle XYZ is self-polar, since the polar line of any vertex is the side opposite that vertex. The tangents

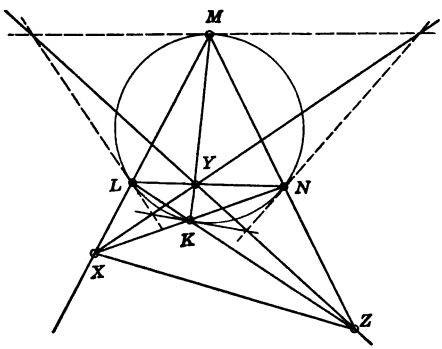


FIG. 55.

at K, L, M, N intersect in pairs on the sides of the triangle XYZ; and hence the sides of XYZ are the diagonals of the complete circumscribed quadrilateral formed by these tangents. For this reason, the triangle XYZ is called a **diagonal triangle.**

Opposite sides of the complete inscribed quadrangle $KLMN$ intersect in the vertices of the diagonal triangle; and hence the diagonal triangle is completely determined by the vertices of the quadrangle. The diagonal triangle is thus self-polar with respect to any conic passing through K, L, M, and N.

Any two points X and Z, conjugate with respect to a conic, are vertices of a diagonal triangle; the third vertex is the intersection of the polar lines of X and Z.

72. Pole-rays and Polar Planes with Respect to a Cone.—If we project a conic from a point not lying in its plane we obtain a cone; and the properties we have been developing with reference to the conic go over into properties connected with the cone. Thus, by this projection, pole and polar line with respect to the

conic become **pole-ray** and **polar plane** with respect to the cone. If a is any line through the vertex of the cone, and we draw any plane through a cutting the cone in the rays b and d and the polar plane of a in c, then $abcd$ is a harmonic pencil of rays.

The projector of a pair of polar figures with respect to the conic is a pair of polar figures with respect to the cone.

The projector of a diagonal triangle is a **diagonal pyramid.** The faces of a diagonal pyramid are the diagonal planes of a complete 4-face circumscribed about the cone; and the edges of a diagonal pyramid are the intersections of pairs of opposite faces of a complete inscribed 4-edge.

If a plane touches a cone, its pole-ray is the line of contact of the plane; and the polar plane of a ray of the cone is the tangent plane along that ray.

A cone may be thought of as polarizing the bundle of rays whose center is its vertex in such a way that to every ray of the bundle corresponds a definite plane (polar plane) and to every plane of the bundle corresponds a definite ray (pole-ray).

Exercises

1. Construct the polar figure of a given triangle one of whose sides is the ideal line.

2. If two tangents to a conic vary so that their chord of contact envelopes a second conic, show that their intersection will trace a third conic; and conversely.

3. Two conics intersect in four points, construct the diagonal triangle common to both.

4. Two conics intersect in two points, construct one vertex and the side opposite of the diagonal triangle common to both.

5. If two conics do not intersect each other, construct the diagonal triangle common to both.

6. Given two conics in the plane. Any point in the plane, as A, has a polar line with respect to each of them, and if these polar lines intersect in A_1, A and A_1 are conjugate with respect to both curves. Show that if A describes a straight line, A_1 will, in general, describe a third conic which passes through the vertices of the diagonal triangle common to the two given conics.

NOTE.—The theory of poles and polar lines is due to Desargues,

whose development of the theory is contained in his *Brouillon projet d'une atteint*—(1639). Earlier writers had discovered certain theorems and properties. Apollonius, for example, knew that the intersection of two tangents to a conic is harmonically separated from the chord of contact by the curve points on any secant through the intersection.

The correlation in exercise 6 is known as a Steiner Correlation. It belongs to the theory of quadric transformations.

CHAPTER IX

DIAMETERS, AXES, AND ALGEBRAIC EQUATIONS OF CURVES OF SECOND ORDER

73. Diameters and Centers of Conics.—The theory of poles and polar lines with respect to a conic leads to important metric properties of the conic. Thus, the polar line of an ideal point is a **diameter** of the conic; and the pole of the ideal line is the **center** of the conic.

Every diameter of a conic passes through the center (Art. 69); and the segment between the curve points on each diameter is bisected by the center (Art. 64). Hence (Art. 65), the center of an ellipse is inside the curve; the center of a hyperbola is outside the curve; and the center of a parabola is the point of contact of the curve with the ideal line. The diameters of a parabola, therefore, form a system of parallel lines.

Again (Art. 64): *Any diameter of a conic bisects a system of parallel chords; namely, all the chords which, when extended, pass through the infinitely distant pole of the diameter.*

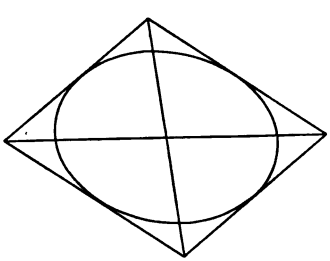


FIG. 56.

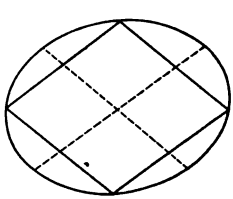


FIG. 57.

74. Conjugate Diameters.—Two diameters of a conic, each of which passes through the pole of the other, are called **conjugate diameters** with respect to the conic. Hence, by the preceding article: *Each of two conjugate diameters bisects all the chords of the conic drawn parallel to the other.*

A pair of conjugate diameters together with the ideal line form the sides of a diagonal triangle (Art. 71). It follows from the definition of a diagonal triangle that: The diagonals of *a parallelogram circumscribed about a conic are conjugate diameters with respect to the conic* (Fig. 56).

Again: *The sides of a parallelogram inscribed in a conic are parallel to a pair of conjugate diameters* (Fig. 57).

It follows at once that, if any point on a conic is joined to the extremities of a diameter, the two chords so formed are parallel to a pair of conjugate diameters.

Exercises

1. Suppose a conic is given by four points and the tangent at one of them, construct a diameter and determine the center.

2. Draw the chord of a given conic which is bisected at a given point.

3. Prove that the chords of a given conic which are bisected by any given chord envelope a parabola.

Suggestion.—Use 1, Art. 69, to show that the given chord is projectively related to the ideal line in such a way that any bisected chord joins corresponding points.

4. Construct a conic having given:

(*a*) Two points and one pair of conjugate diameters;
(*b*) Two tangents and one pair of conjugate diameters;
(*c*) Three points and the center;
(*d*) Three tangents and the center;
(*e*) One point and two pairs of conjugate diameters;
(*f*) One tangent and two pairs of conjugate diameters.

5. Construct a parabola having given either three points or three tangents and the direction of its diameters.

75. Application of the Harmonic Properties of Poles and Polar Lines.—As special cases of the general definitions and theorems in Art. 64, we have the following propositions:

1. *A pair of conjugate diameters of a hyperbola is harmonically separated by the asymptotes.*

For the pencil in question consists of two tangents and a pair of conjugate lines drawn from the intersection of the tangents.

2. *The chord determined upon any secant of a hyperbola is bisected by the diameter conjugate to the secant* (Art. 73).

By 1, the segment of the secant contained between the asymptotes is also bisected by the conjugate diameter. Hence:

3. *The two segments of any secant which lie between a hyperbola and its asymptotes are equal in length.*

This proposition furnishes a neat construction of a hyperbola when the asymptotes and one point of the curve are given. Thus (Fig. 58), let OA and OA_1 be the given asymptotes, and P

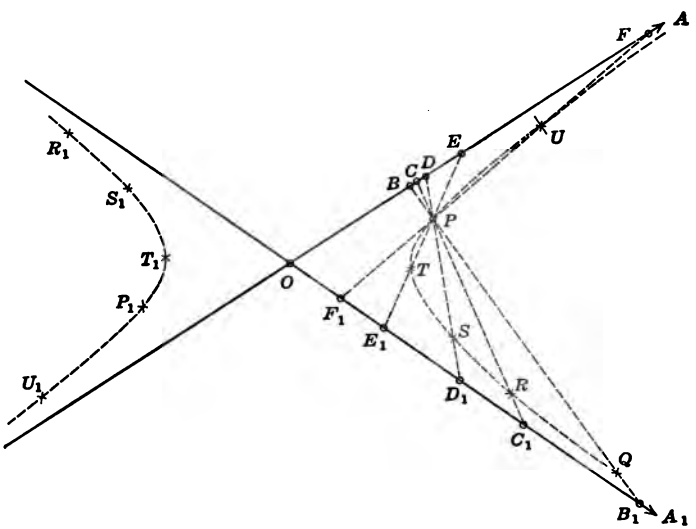


FIG. 58.

be any point on the curve. Draw a series of rays through P meeting OA in B, C, D,——, and OA_1 in B_1, C_1, D_1,——. Lay off segments B_1Q, C_1R, D_1S,—— equal respectively to the segments BP, CP, DP,——. Then the points Q, R, S,—— are points on the hyperbola. Since O is the center of the curve, the points P_1, Q_1, R_1, S_1,——, symmetrical to P, Q, R, S,—— with respect to O, also lie on the curve.

4. *The segment of any tangent to a hyperbola contained between the asymptotes is bisected by the point of contact.*

For, if one of two conjugate diameters is parallel to a tangent, the other passes through the point of contact.

5. *The segment of the line contained between the pole of any chord of a parabola and the mid-point of the chord is bisected by the curve.*

For the line in question is a diameter of the parabola, and the two curve points on it harmonically separate the pole from the mid-point of the chord.

Exercises

1. Given five points, two of which are ideal; construct the asymptotes of the hyperbola determined by the given points. Construct a series of points of the curve.

2. Given the asymptotes of a hyperbola and one other tangent; construct a series of points of the curve.

3. Given five points in a plane; construct a pair of conjugate diameters of the curve determined by the given points.

4. Given five lines in a plane; construct the center of the curve to which the five lines are tangent.

5. Show that the perpendiculars dropped from any point S upon the diameters of a given conic meet the conjugate diameters in points of a hyperbola which passes through S and through the center of the given conic.

76. The Axes of a Conic.—We have seen (Art. 73) that a diameter of a conic bisects a system of parallel chords. Each of these

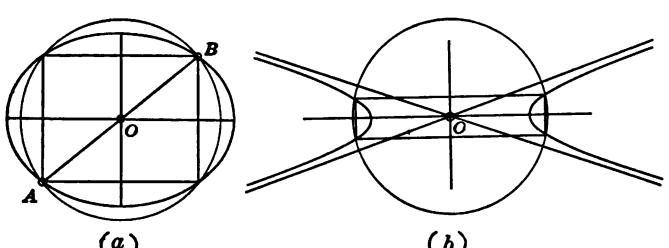


FIG. 59.

chords is conjugate to the diameter, since each passes through the pole of the diameter. Any diameter, therefore, has a system of parallel conjugate chords.

If a diameter is perpendicular to its system of parallel chords, the diameter is called an **axis** of the conic.

The following considerations establish the existence of at least

one axis for each conic and exhibit the method for constructing axes when the curve is fully drawn.

First, suppose the conic is an ellipse or a hyperbola (Fig. 59). Construct the center O, the pole of the ideal line. The diameters of the given conic are evidently not all of the same length, for then the conic would be a circle. Let AB be a diameter which is neither the shortest nor the longest of the diameters. The circle on AB as diameter must then meet the given conic in four points

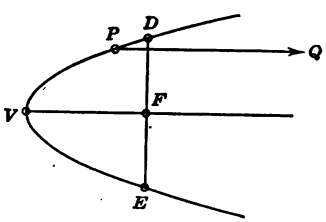


FIG. 60.

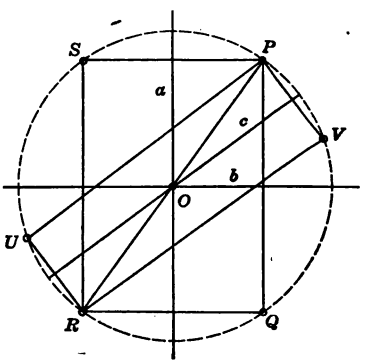


FIG. 61.

of which A and B are two. We can now construct a rectangle inscribed in the given conic. The diameters of the conic parallel to the sides of this rectangle are axes.

Second, suppose the given conic is a parabola (Fig. 60). Construct any diameter, as PQ, the polar line of an ideal point. Draw the chord DE perpendicular to PQ and let F be its mid-point. The line VF, parallel to PQ, is an axis.

It thus appears that an ellipse, or a hyperbola, has at least two axes perpendicular to each other; and a parabola has at least one axis.

We can now show that an ellipse or a hyperbola cannot have more than two axes; and a parabola cannot have more than one axis. For, if a and b are two axes of an ellipse or of a hyperbola (Fig. 61), and if P is any point of the curve, we can construct the rectangle $PQRS$ inscribed in the curve. If c is a third axis, we can also construct the rectangle $PVRU$ inscribed in the curve. But

the circle whose center is O and whose radius is OP passes through the six points P, V, Q, R, U, S and consequently has six points in common with the given curve. The circle, therefore, coincides with the given conic (Art. 46). In other words, if a conic has more than two axes, it is necessarily a circle, and then every diameter is an axis.

In Fig. 62, let a be an axis of a parabola and P be any point of the curve. Then a must bisect the perpendicular chord PR. If c is also an axis, c is parallel to a and must bisect the perpendicular chord PS. Thus the straight line PR meets the parabola in the three points P, S, R. But this is impossible. Hence, a parabola can have but one axis.

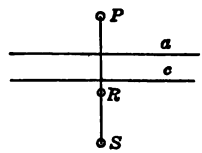


FIG. 62.

In conclusion, we have shown that:

An ellipse, or a hyperbola, has exactly two mutually perpendicular axes; and a parabola has but one axis.

It follows, from the construction, that the axes of an ellipse, or of a hyperbola, are conjugate diameters of the conic.

77. The Vertices of a Conic.—The points in which an axis meets a conic are called **vertices** of the conic.

An ellipse has four vertices since every diameter, and consequently each axis, meets the curve in two points.

A hyperbola has but two vertices since its center lies outside the curve and any two conjugate diameters, and consequently the axes, are separated harmonically by the asymptotes (Art. 75, 1). Thus, but one of the axes meets the curve.

A parabola has but one actual vertex since every diameter, and therefore the axis, meets the curve in its ideal center.

Exercises

1. Construct a series of points on a parabola, having given two points or two tangents and the axis.

2. Given four tangents to a parabola; construct its axis and vertex.

3. Construct a series of points on a conic, having given three points or three tangents and one axis.

4. Through a point S in the plane of a given conic lines are drawn parallel to the diameters of the conic. Show that the sheaf S is projectively related to the sheaf of conjugate diameters, the two sheaves

generating a second conic. How are the axes of the second conic related to the axes of the given conic?

5. Show that the axes of an hyperbola bisect the angles formed by the asymptotes.

78. Algebraic Equations of the Conics.—Algebraic equations for the several conics are derived from special cases of the quadrangle theorem or its dual (Art. 57).

1. *The Hyperbola.*—In Fig. 63, let the asymptotes be the coördinate axes, and let P be any point on the curve whose coördinates are $x = OQ$ and $y = QP$. The tangent at P meets the asymptotes in K and N, and the segment KN is bisected at P (Art. 75, 4). Hence $ON = 2x$ and $OK = 2y$.

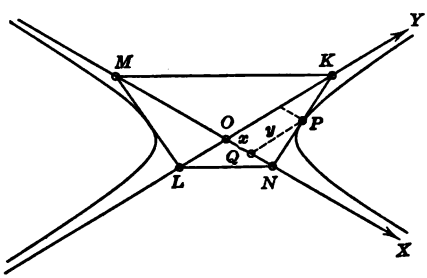


FIG. 63.

If LM is a second tangent meeting the asymptotes in L and M, the quadrangle $KLMN$ circumscribes the curve, and its diagonals, KM and LN, are parallel since they must meet on the ideal line (Art. 57). The two triangles LKN and NLM are therefore of equal area. If from each of these triangles we subtract their common part, namely, the triangle LON, we obtain the triangles KON and LOM also equal in area. But the area of KON is

$$\tfrac{1}{2}\, ON \cdot OK \sin NOK = 2xy \sin NOK.$$

For all positions of P this area is constantly equal to the area of the fixed triangle LOM. Therefore:

The algebraic equation of a hyperbola, referred to its asymptotes as coördinate axes, is $xy = k$, k being a constant.

2. *The Ellipse.*—Choose for coördinate axes a pair of conjugate diameters, as AB and QR (Fig. 64). Let P be any point on the

ellipse whose coördinates are $x = OD$ and $y = DP$. The tangents at the opposite vertices of the inscribed quadrangle $APBQ$ must meet on the line ST (Art. 57). But the tangents at A and B are parallel to QR. Hence ST is parallel to QR. We have, then, a series of similar triangles from which we derive,

$$\frac{AD}{DP} = \frac{AH}{HS}, \qquad \frac{DB}{DP} = \frac{HB}{HT}, \tag{1}$$

and

$$\frac{AH}{HT} = \frac{AO}{OQ} = \frac{OB}{OQ} = \frac{HB}{HS}. \tag{2}$$

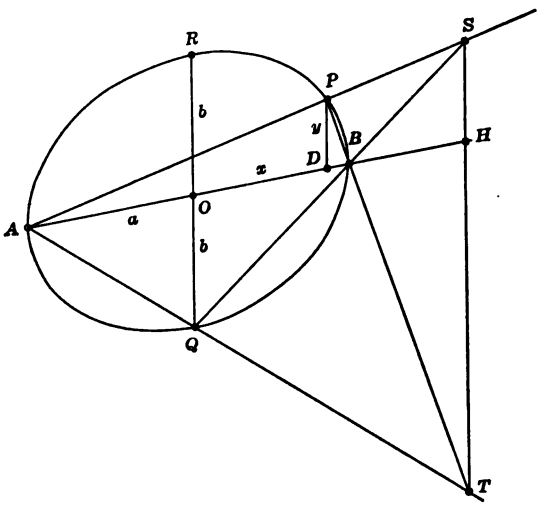


FIG. 64.

If the lengths of the semi-diameters AO and OQ are represented by a and b respectively, we have from (1),

$$\frac{a+x}{y} = \frac{AH}{HS}, \quad \frac{a-x}{y} = \frac{HB}{HT}, \tag{3}$$

and from (2),

$$\frac{AH}{HT} = \frac{HB}{HS} = \frac{a}{b}.$$

Hence, from (3) by multiplication,

$$\frac{a^2 - x^2}{y^2} = \frac{AH}{HS} \cdot \frac{HB}{HT} = \frac{a^2}{b^2},$$

from which we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Therefore: *The equation of an ellipse, referred to a pair of conjugate diameters as coördinate axes, is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are the lengths of the semi-diameters.

In this connection, it is worth noting that, as P describes the ellipse, the points S and T describe point-rows along the fixed lines

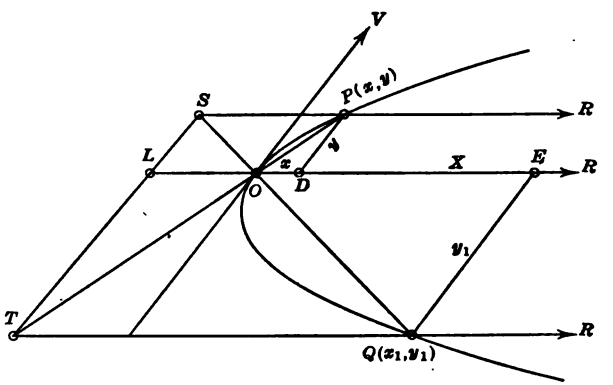


FIG. 65.

BQ and QA in perspective position, since the line ST is constantly parallel to QR. The sheaves of rays described by AS and BT are, therefore, projectively related and generate the ellipse. The chain of perspectivity

$$A \overline{\wedge} BQ \overline{\wedge} QA \overline{\wedge} B$$

furnishes a simple construction for the ellipse when the lengths and the position of a pair of conjugate diameters are known.

The circle is a special case of an ellipse in which the semi-diameters are of equal length. The equation of the circle, referred to a pair of diameters as coördinate axes, is therefore

$$x^2 + y^2 = a^2.$$

3. *The Parabola.*—Choose any diameter as OX for the X-axis; and the tangent, OY, at its extremity for the Y-axis (Fig. 65). Let $P(x, y)$ and $Q(x_1, y_1)$ be any two points on the curve, and R the infinitely distant point on OX. The quadrangle $QOPR$ is then inscribed in the curve. The tangents at O and R must meet on the line ST (Art. 57). Thus ST is parallel to OY. Also SP and TQ are parallel lines. Hence,

$$\frac{y}{x} = \frac{DP}{OD} = \frac{LT}{LO} = \frac{EQ}{LO} = \frac{y_1}{LO} \tag{1}$$

and

$$\frac{y_1}{x_1} = \frac{EQ}{OE} = \frac{LS}{LO} = \frac{DP}{LO} = \frac{y}{LO} \tag{2}$$

Eliminating LO from (1) and (2) we have

$$\frac{y^2}{x} = \frac{y_1^2}{x_1}.$$

The ratio y^2/x is, therefore, the same for all points on the parabola. Hence: *The equation of a parabola, referred to any diameter and the tangent at its extremity as coördinate axes, is*

$$y^2 = kx,$$

where k is a constant.

The parabola is completely determined by three fixed points Q, O, R and the tangents at O and R; and is generated by projectively related sheaves of rays with centers at O and R, corresponding rays meeting in the various positions of P. This remark furnishes an easy construction for the parabola.

The results of this article establish completely the identity of the curves of second order with the conics of analytic geometry.

Exercises

1. Construct an ellipse according to the suggestions made in 2 of the preceding article.

2. Construct a parabola according to the suggestions in 3 of the preceding article.

3. Let u and u_1 be a pair of parallel lines and AB any transversal cutting u in A and u_1 in B; let TS be a second transversal, inclined to the first, and meeting u in T and u_1 in S. If TS moves parallel to itself, it cuts u and u_1 in point-rows in perspective position. Show that the sheaves of rays described by AS and by BT are projectively

related and generate a hyperbola of which AB is a diameter; that the conjugate diameter is parallel to TS; and, if $AB = 2a$ and $TS = 2b$, the equation of the hyperbola, referred to these diameters as coördinate axes, is $x^2/a^2 - y^2/b^2 = 1$, AB being the X-axis.

79. Diametral Planes and Axes of Cylinders.—The properties of conics which depend upon the Pascal theorem or the Brianchon theorem have been, thus far, transferable to the cone by simple projection. For example, we have seen (Art. 72) that the theory of poles and polar lines with respect to a given conic can be transferred at once by projection into a like polar theory with respect to a given cone in the bundle of rays whose center is the vertex of the cone.

The properties of diameters and axes are only special cases of the properties of poles and polar lines with respect to a fixed conic. But when we attempt to develop corresponding properties for the cone by means of projection, a difficulty at once arises because the ideal plane does not pass through the vertex of the cone unless the vertex is infinitely distant; that is, unless the cone is a cylinder. For the cylinder, then, we have a theory of diametral planes and axes derivable by projection from the properties of the diameters and axes of the conic of which the cylinder is a projector. Thus, the pole-ray of the ideal plane is the **axis** of the cylinder. The axis is the projector of the center of the conic. The axis of a parabolic cylinder is an ideal line; the axis of an elliptic cylinder or of an hyperbolic cylinder, is an actual line.

Any plane through the axis of a cylinder is a **diametral plane.** The pole-ray of a diametral plane is an ideal line. A diametral plane is the projector of a diameter of the conic of which the cylinder is a projector. Two diametral planes, each containing the pole-ray of the other, are **conjugate** diametral planes.

Any plane perpendicular to the axis of an elliptic cylinder, or an hyperbolic cylinder, cuts the cylinder in a conic. The axes of this conic, together with the axis of the cylinder, determine a pair of diametral planes perpendicular to each other. These planes are the **principal planes** of the cylinder. Any plane parallel to one of the principal planes cuts the cylinder in a pair of lines equidistant from the other principal plane. The principal planes are, therefore, **planes of symmetry.**

CHAPTER X

RULED SURFACES OF SECOND ORDER

80. Ruled Surfaces.—When a straight line moves continuously in space, it describes, or generates, a **ruled surface.** The form of the surface depends upon the law governing the motion of the line. Thus, if the line passes through a fixed point and meets a fixed curve, it describes, or generates, a cone. If the fixed point is ideal, the cone generated is a cylinder; if the fixed curve is a straight line, the surface generated is a plane.

When a point describes continuously one of two projectively related point-rows which generate a regulus, its corresponding point describes continuously the other point-row (Art. 39, theorem VI), and the line joining corresponding points describes continuously the regulus. The rays of a regulus thus lie on a smooth surface which is called a **ruled surface of second order.** Each ray of the regulus is called a **generator** of the surface.

The following theorem is fundamental for the ruled surfaces of second order.

Theorem XII.—*A surface of second order upon which lies a regulus V contains a second regulus U. Each ray of either regulus meets all the rays of the other.*

Let u and u_1 be the two projectively related point-rows which generate the regulus V (Fig. 66), the lines v, v_1, v_2, v_3,— being rays of V. A sheaf of planes having any one of these rays as an axis cuts out projectively related point-rows along any two of the other rays. For example, the sheaf of planes whose axis is v_1 cuts out projectively related point-rows along v and v_2. The lines joining corresponding points on v and v_2 constitute rays of a second regulus U to which belong the lines u and u_1. Every ray of U meets the three rays v, v_1, v_2 and, in consequence, must meet all the rays of V. For the two sheaves of planes determined by u_2 and the two point-rows u and u_1 are coaxial and projectively

related; and the three planes u_2A, u_2B, and u_2C coincide respectively with their corresponding planes u_2A_1, u_2B_1, u_2C_1. The two sheaves thus have three self-corresponding planes and consequently coincide throughout (theorem VII). Since the plane u_2D coincides with the plane u_2D_1, u_2 meets v_3; and similarly must meet the other rays of V. In the same way we can show that the rays u_3, u_4,—— must meet all the rays of V. The surface that contains the regulus V, therefore also contains the regulus U.

No two rays of either regulus ever meet each other, for then all the rays of that regulus would be coplanar; but any ray of either regulus meets all the rays of the other.

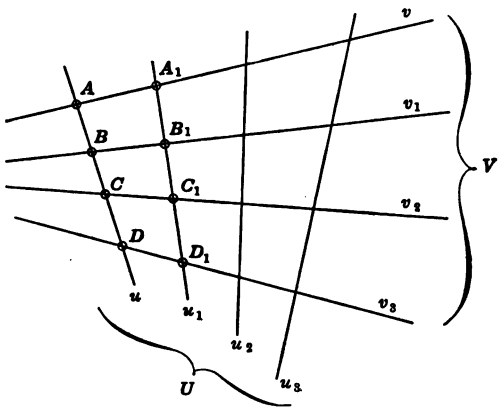


FIG. 66.

Through any point on a ruled surface of second order there pass two rays, one from each regulus lying on the surface.

A straight line, not lying wholly on the surface, cannot meet the surface in more than two points. For, if a line has three points in common with the surface, it meets three rays of one regulus, and consequently all the rays of that regulus, and so must belong to the other regulus. It is for this reason that the surface is said to be of **second order.**

81. Sections of a Surface of Second Order.—*A surface of second order is cut by any plane, not containing a ray of either regulus, in a conic.*

For the cutting plane meets either regulus in a conic (Art. 47).

If the cutting plane contains a ray of either regulus, we have the following theorem:

Any plane which passes through a ray of either regulus contains a ray of the other regulus and does not meet the surface outside these rays.

For, suppose the plane α passes through the ray v_i of the regulus V and meets two other rays of the same regulus in the points M and N (Fig. 67). The straight line MN meets three rays of the regulus V and must, therefore, meet all of them. It is then a ray of the other regulus U, say the ray u_j. Hence α has the two lines v_i and u_j in common with the surface. That it can have no point outside these lines in common with the surface

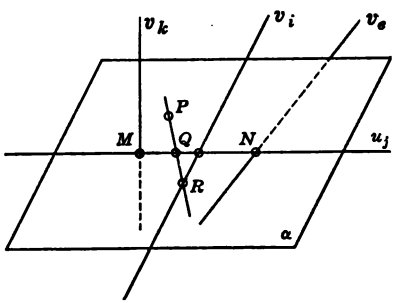


FIG. 67.

follows in this way; suppose there is a point P, not lying on either v_i or u_j, but which is common to the plane and to the surface. We can then draw any number of lines through P meeting both v_i and u_j and therefore, having three points in common with the surface. But this is impossible unless all these lines lie entirely on the surface (Art. 80), in which case the entire plane must form a part of the surface. But a plane cannot form part of the surface since no two rays of either regulus lie in any plane. We conclude, therefore, that α cannot meet the surface outside the two lines v_i and u_j. The plane α can be any plane through any ray of either regulus.

82. Tangent Lines and Tangent Planes.—*A plane which contains a ray of either regulus on a ruled surface of second order*

is a tangent plane to the surface, the point of contact being the intersection of the two rays lying in the plane.

For all the lines which pass through the intersection of the two rays and lie in the plane cannot meet the surface outside this point of intersection. These lines are, therefore, tangent lines to the surface at the point of intersection; and the plane containing them is the tangent plane to the surface at that point.

83. Tangent Cones.—If we project a ruled surface from a point S not lying on the surface, we obtain a sheaf of planes of second class. For the projector of a regulus, from any point not on a ray of the regulus, is a sheaf of planes of second class (Art. 47). Each plane of the sheaf contains two generators of the surface, one from each regulus, and is therefore, tangent to the surface at the point of intersection of the generators. The sheaf of planes envelopes a cone whose center is S. The cone is the **tangent cone** to the surface from S. The line joining S to the point of contact on any plane of the sheaf of planes is a **ray** of the tangent cone. The lines drawn from the eye touching the surface are rays of the tangent cone whose vertex is the eye.

84. Polar Planes.—*The points of contact on the rays of a tangent cone lie in one and the same plane.*

No three of the points of contact can lie on the same straight line, for then this line must lie entirely on the surface and the tangent planes at the three points must coincide with the plane determined by this line and the vertex of the cone; and the rays of one regulus which pass through the three points must lie in the plane so determined. But this is impossible. Hence, three of the points of contact determine a plane which cuts the surface in a conic and also cuts the tangent cone in a conic. But these two conics coincide throughout, since they have three points and the tangents at them in common (Art. 46).

If A is the vertex of a tangent cone and α is the plane containing the points of contact on the rays of the cone, α is called the **polar plane** of A with respect to the surface.

The surface-points on any line through A are separated harmonically by A and the polar plane of A.

For any plane through A, which is not a tangent plane, meets the surface in a conic, and the polar plane of A in the chord of contact of tangents to the conic from A.

85. Circumscribing Tetrahedrons.—Let P and P_1 be any two points on a surface of second order; a and b_1, the two generators passing through P; and a_1 and b, the two generators through P_1 (Fig. 68). Since each ray of either regulus meets all the rays of the other, it follows that a meets b in a point Q and a_1 meets b_1 in a point Q_1. Each face of the tetrahedron PP_1QQ_1 is a tangent plane to the surface, the point of contact being a vertex. The tetrahedron thus formed circumscribes the surface and is completely determined by the pair of points P, P_1.

86. The Class of a Ruled Surface of Second Order.—The **class** of a surface is defined as the maximum number of tangent planes that can be drawn through a straight line not lying entirely on the surface.

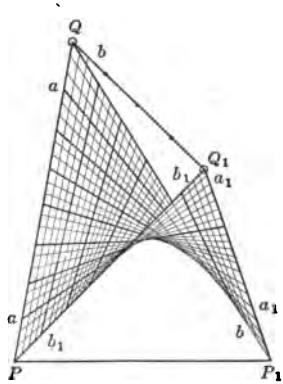


FIG. 68.

A ruled surface of second order is of second class.

For a line, not lying entirely on the surface, cannot meet the surface in more than two points; and but two of the faces of the circumscribing tetrahedron, determined by these points, contain the line.

If a line touches the surface, but one tangent plane can be drawn through the line, namely, the plane determined by the generators of the surface through the point of contact of the line.

If a line does not meet the surface, no tangent plane can be drawn through the line; for the line does not then meet any of the generators of the surface.

If a line lies on the surface, every plane through it is tangent to the surface.

We conclude, therefore: *As many tangent planes to a surface of second order can be drawn through a given line as the line has points in common with the surface.*

87. Classification of Ruled Surfaces of Second Order.—We have seen that a plane may meet a ruled surface of second order in a conic or in two straight lines, and in the latter case, that the

plane is tangent to the surface (Art. 81). If the ideal plane meets the surface in a conic, the surface is called a **hyperboloid of one sheet** or a **unapartite hyperboloid** (Fig. 69). If the ideal plane is tangent to the surface, that is, if it contains a ray of each regulus the surface is called a **hyperbolic paraboloid** (Fig. 70).

The hyperboloid of one sheet is cut by an arbitrary plane in an ellipse, a parabola, or a hyperbola, according as the cutting

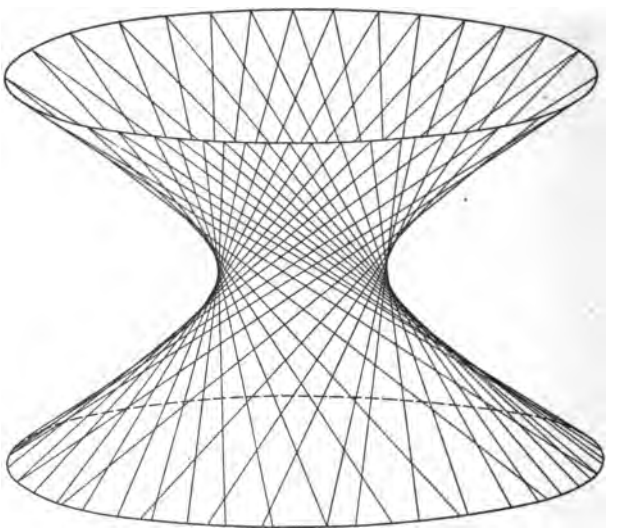


FIG. 69.

plane meets the infinitely distant conic in no point, in one point, or in two points.

The hyperbolic paraboloid is cut by an arbitrary plane in a hyperbola, or a parabola, according as the cutting plane meets the infinitely distant generators in two distinct points, or passes through their point of intersection.

The tangent cone to a hyperboloid of one sheet, whose points of contact lie on the infinitely distant conic, is the **asymptotic cone.** The vertex of the asymptotic cone is the **center** of the surface. The ideal plane is the polar plane of the center.

Exercises

1. Show that the locus of the vertex of a cone of second order, to which the six sides of a twisted hexagon are tangent, is a ruled surface of second order of which the three principal diagonals of the hexagon are generators.

2. Show that the three principal diagonals of a twisted hexagon, whose six sides lie upon a ruled surface of second order, intersect in one point.

3. A point-row u and a sheaf of rays S are projectively related but do not lie in parallel planes. Show that lines drawn through the points of u parallel to the corresponding rays of S constitute one regulus of a hyperbolic paraboloid.

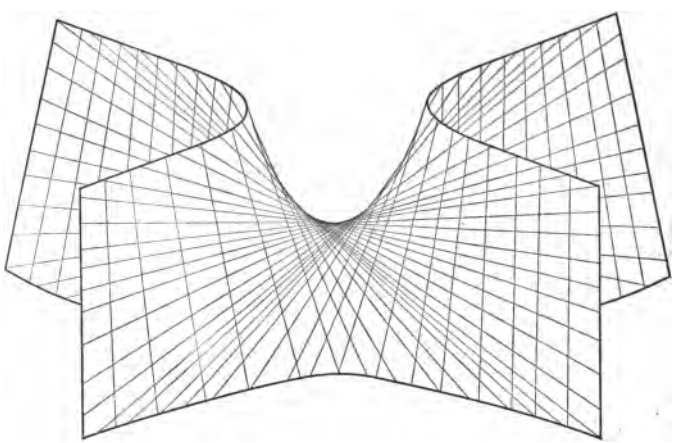


FIG. 70.

4. A point-row u is projectively related to a sheaf of planes v, and u is not perpendicular to the axis of v. Show that lines drawn through the points of u perpendicular to the corresponding planes of v form one regulus of a hyperbolic paraboloid.

5. Perpendiculars to a surface of second order are erected at the points of a line lying upon the surface. Show that these perpendiculars form one regulus of a hyperbolic paraboloid. Show also that the generators of the paraboloid are parallel to a pair of perpendicular planes. Such a paraboloid is said to be *equilateral*.

6. Show that lines drawn through any point parallel to the rays of a regulus lie in one plane or upon a cone, according as the regulus belongs to a hyperbolic paraboloid or to a hyperboloid of one sheet.

CHAPTER XI

PROJECTIVELY RELATED ELEMENTARY FORMS

88. Four Harmonic Elements of an Elementary Form.—An elementary form is generated by two projectively related primitive forms and any four harmonic elements of either of these primitive forms determine, with their corresponding elements, **four harmonic elements** of the elementary form. For example, four points of a conic are harmonic whenever they are the intersections of corresponding rays in two harmonic pencils belonging to the sheaves of rays which generate the conic. Since a conic is projected from any two of its points in projectively related sheaves of rays (Art. 45, 3), four points of a conic are harmonic whenever they are projected from any other point of the curve in a harmonic pencil of rays.

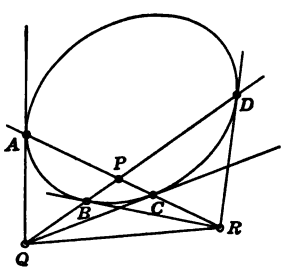


FIG. 71.

If a conic is cut by two conjugate lines AC and BD (Fig. 71), *the points of intersection A, B, C, D are four harmonic points of the conic.*

For, if Q is the pole of AC and R is the pole of BD, then $QBPD$ and $APCR$ are harmonic ranges (Art. 64). Hence we have,

$$A(QBPD\text{——}) \overline{\wedge} D(APCR\text{——});$$

corresponding rays intersecting in the points of the conic A, B, C, D,——. Therefore, the points A, B, C, D are harmonic by definition.

Conversely: *If the four points A, B, C, D of a conic are harmonic points, AC and BD are conjugate lines with respect to the conic.*

For the pencil formed by the lines DA, DB, DC, and the tangent at D is harmonic; and the line AC cuts this pencil in the

harmonic range $APCR$. Hence AC passes through the pole of BD.

In general, four elements of an elementary form are harmonic whenever they are projected from, or cut by, any fifth element of the form in four harmonic elements of a primitive form.

Thus: *The tangents at four harmonic points of a conic form four harmonic rays of the envelope of second class whose rays are tangents to the conic.*

For the pencils $Q(APCR)$ and $R(QBPD)$ (Fig. 71) are harmonic and cut the tangents at B and C, respectively, in harmonic ranges. Pairs of corresponding points on these ranges are joined by the tangents at A, B, C, D.

Four rays of a cone are harmonic if they are projected from any fifth ray of the cone in a harmonic pencil of planes; four planes of a sheaf of planes of second class are harmonic if they are cut by any fifth plane of the sheaf in a harmonic pencil of rays.

The above general definition does not apply to a regulus, since four rays of a regulus are never cut by any fifth ray of the same regulus. Four rays of a regulus are harmonic whenever they are cut in a harmonic range by any ray of the other regulus on the same surface of second order. If four rays of a regulus are cut in a harmonic range by any ray of the other regulus, they are cut in harmonic ranges by all rays of the other regulus.

The definition of four harmonic elements of a primitive form (Art. 21) is thus extended to include the elementary forms.

89. Elementary Forms in Perspective Positions with Primitive Forms.—An elementary form is **in perspective position** with either of two primitive forms which generate it. A conic is in perspective position with a sheaf of rays whose center is any point of the conic, and whose rays pass through their corresponding points of the conic; an envelope of second class is in perspective position with the point-row it cuts upon any one of its rays; a cone is in perspective position with a sheaf of planes whose axis is any one of its rays, the planes of the sheaf containing their corresponding rays of the cone; a regulus is in perspective position with the point-row it cuts upon any ray of the other regulus on the same surface of second order. The primitive form, in these examples, may be regarded as one of two pro-

jectively related primitive forms which generate the elementary form.

90. Elementary Forms in Perspective Positions with Each Other.—Two correlated elementary forms, consisting of like elements, are in perspective position with each other, if each is a section of, or projector of, the same third elementary form. Thus, two conics are in perspective position with each other if each is a section of the same cone, or of the same regulus.

Two correlated elementary forms, consisting of unlike elements, are in perspective position with each other, if each element of one lies on, or passes through, the corresponding element of the the other. Thus, a cone and a conic are in perspective position, if each point of the conic lies on the corresponding ray of the cone; a regulus is in perspective position with a sheaf of planes of second class, if each ray of the regulus lies in the corresponding plane of the sheaf.

The concept of perspective position, defined for primitive forms (Art. 15), is thus extended to include the elementary forms.

We shall often speak of the primitive forms and the elementary forms together as **forms**; and we may obviously consider chains of perspectivity (Art. 34) which may include elementary forms as well as primitive forms.

Two forms in a chain of perspectivity are correlated so that to any four harmonic elements of either always correspond four harmonic elements of the other.

91. Projectively Related Forms.—We are now in a position to extend the definition of projective relationship (Art. 35) so as to include the elementary forms. Thus, two correlated forms are **projectively related** if any four harmonic elements of either correspond to four harmonic elements of the other. For example, two forms are projectively related if they are in perspective position, or if they are contained in a chain of perspectivity.

Two projectively related forms can always be connected by a chain of perspectivity.

Consider, for example, two projectively related cónics k and k_1. Let S be a sheaf of rays in perspective position with k and S_1 a sheaf of rays in perspective position with k_1. S and S_1

can be connected by a chain of perspectivity as in Art. 43. We then have the chain of perspectivity

$$k \overline{\wedge} S \overline{\wedge} S_2 \overline{\wedge} S_1 \overline{\wedge} k_1.$$

When two forms are projectively related, we shall say that a **projectivity** exists between them; and we shall often think of this projectivity as transforming either form into the other, the transformation being effected by a chain of perspectivity connecting the two forms.

92. Determination of Projective Relationship.—Theorem VIII, Art. 43, can now be extended to include the elementary forms. Thus: *A projectivity can always be established between any*

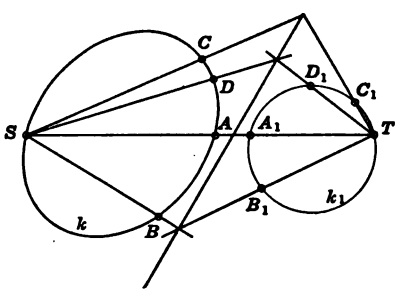


FIG. 72.

two forms so that any three elements of the one shall correspond to three elements of the other chosen arbitrarily.

Thus, for example, to establish a projectivity between the two conics k and k_1 (Fig. 72) so that the points A, B, C of k shall correspond respectively to the points A_1, B_1, C_1, of k_1, join A and A_1 and let the line AA_1 meet k again in S, and k_1 again in T. Project k from S and k_1 from T, and thus obtain two sheaves of rays, S and T, which are projectively related and in perspective position. Hence the chain of perspectivity

$$k \overline{\wedge} S \overline{\wedge} T \overline{\wedge} k_1.$$

If D is any point of k, the corresponding point, D_1 of k_1, can be immediately constructed.

In a similar manner, a chain of perspectivity can be constructed

between any two projectively related forms, having given three pairs of corresponding elements.

Two projectively related elementary forms may be used to generate a new form, provided a pair of corresponding elements determines a third element variable with the pair. Thus, the projectively related conics just considered generate a sheaf of rays of higher class.

93. Superposition of Projectively Related Forms.—Two forms, consisting of like elements, may be superposed as in Art. 40. The two forms then have a common base, or support. Thus, two point-rows of second order are superposed when they lie on the same conic; two envelopes of second class are superposed when they each form the system of tangents to the same conic; two reguli are superposed when they each belong to the same regulus; and so on.

When two projectively related forms are superposed, we shall say that they constitute a **projectivity** upon their common base. Thus, two projectively related and superposed point-rows of second order constitute a projectivity upon their common conic; two projectively related and superposed point-rows of first order form a projectivity upon their common support: and so on.

A projectivity may be established upon any form by choosing arbitrarily three pairs of corresponding elements.

We shall often think of a projectivity upon a form as a transformation which changes the form into itself; the transformation being affected by a chain of perspectivity connecting the two forms which constitute the projectivity.

94. Double Elements of a Projectivity.—An element which corresponds to itself in a projectivity upon any form is called a **self-corresponding element,** or a **double element,** of the projectivity.

Von Staudt's fundamental theorem (Art. 41) can be extended to include the elementary forms, and may be restated as follows:

A projectivity upon any form cannot have more than two double elements, unless every element is a double element.

For, if there are three double elements, the two forms which constitute the projectivity can be projected from, or cut by, the same fourth element of their common base in two projectively

related primitive forms having three self-corresponding elements and consequently coinciding throughout. The forms themselves, therefore, coincide throughout, each element being a double element of the projectivity. Thus, if a projectivity on a conic has three double points, the two point-rows of second order which constitute the projectivity can be projected from the same fourth point of their common conic in two projectively related sheaves of rays having three self-corresponding rays and, therefore, coinciding throughout. Each point of the conic is then a double point of the projectivity.

95. The Axis of a Projectivity on a Conic.—In a projectivity on a conic (Fig. 73), let the points A, B, C correspond respectively to the points A_1, B_1, C_1, so that

$$ABC\text{——}\bar{\wedge}A_1B_1C_1\text{——}.$$

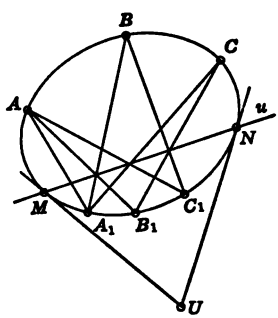


FIG. 73.

From A project $A_1B_1C_1$——, and from A_1 project ABC——. We thus obtain two sheaves of rays in perspective position, since the ray AA_1 corresponds to itself. Corresponding rays in these sheaves meet upon a straight line u, called the **axis** of the projectivity upon the conic. The chain of perspectivity ABC—— $\bar{\bar{\wedge}}A_1(ABC\text{——})\bar{\bar{\wedge}}A(A_1B_1C_1$ ——)$\bar{\bar{\wedge}}A_1B_1C_1$——enables us to construct as many pairs of corresponding points as may be desired. In particular, the points M and N, in which the axis meets the conic, are the double points of the projectivity. Hence:

A projectivity on a conic has two double points, one double point, or no double points, according as the axis of the projectivity meets the conic in two points, one point, or no points.

If we consider the inscribed hexagon $AB_1CA_1BC_1$, we see that the lines BC_1 and B_1C also meet on the axis u. We should then have obtained the same axis had we chosen B and B_1, or C and C_1 as centers from which to project the two point-rows $A_1B_1C_1$—— and ABC——. In other words, the axis is the Pascal line (Art. 53) belonging to the hexagon $AB_1CA_1BC_1$.

We can construct the axis of a projectivity on a conic by

joining pairs of points "crosswise." Thus, the pairs of lines AB_1, A_1B; AC_1, A_1C; BC_1, B_1C meet on the axis. These pairs of lines can be most easily recognized, perhaps, when the points of one point-row are placed over the corresponding points of the other point-row, thus:

$$A\ B\ C\text{——}$$
$$A_1B_1C_1\text{——}$$

96. The Center of a Projectivity on a Conic.—The center of a projectivity on a conic is the pole of the axis with respect to the conic. Thus, in Fig. 73, U is the center of the projectivity determined by the point-rows ABC——and $A_1B_1C_1$—— where U is the pole of the axis u with respect to the conic.

Since the tangents at A, B, C,—— form an envelope of second class projectively related to the point-row ABC—— (Art. 88), it

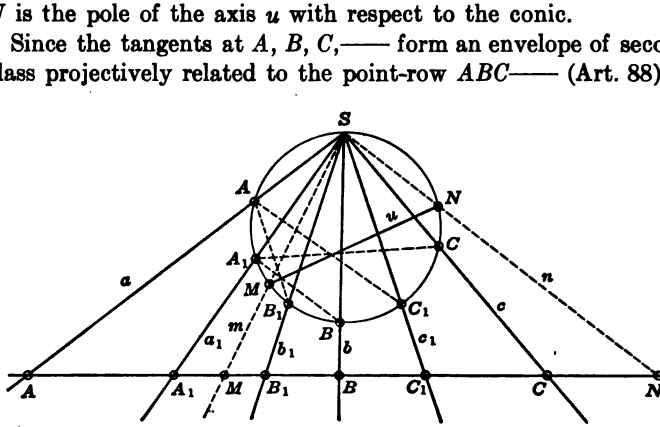


FIG. 74.

follows that the two envelopes, formed by the tangents at $A, B, C,$ ——and the tangents at $A_1, B_1, C_1,$——, are projectively related to each other and thus constitute a projectivity on the envelope formed by all the tangents to the conic. The tangents at the pairs of corresponding points, A, A_1, B, B_1 and C, C_1, are opposite sides of a circumscribed hexagon whose Brianchon point (Art. 53) is U (*cf.* also Art. 70, Fig. 54).

A projectivity on a conic has two double points, one double point, or no double points, according as the center of the projectivity lies outside the conic, on the conic, or inside the conic.

97. Double Elements of a Projectivity on Any Form.—The problem of constructing the double elements of a projectivity on any form can be reduced by projection or section to the problem of constructing the double points of a projectivity on a conic. For example, suppose a projectivity on a sheaf of rays of first class is determined by the three pairs of corresponding rays a, a_1; b, b_1; and c, c_1 (Fig. 74). Cut the sheaf by a conic passing through the center S (for convenience, a circle), and thus obtain a projectivity upon the conic. The double points, M and N, of the projectivity on the conic determine the double rays, m and n, of the projectivity on the sheaf S.

A projectivity on a point-row of the first order can be projected from any point S in a projectivity on the sheaf whose center is S. The construction of the double points of the given projectivity then follows as above. This is illustrated in Fig. 74.

A projectivity on an envelope of second class can be cut by any ray of the envelope in a projectivity on that ray. The double rays of the given projectivity can then be constructed as above.

A projectivity on a cone can be cut by a plane in a projectivity on a conic. The double rays of the given projectivity are determined by the double points of the projectivity on the conic.

98. Application of the Theorem in Art. 95.—We can now solve the following dual problems:

To construct the points in which a given line u meets a conic when the conic is given by five conditions; viz., by five points, or by four points and the tangent at one of them, or by three points and the tangents at two of them.

To construct the tangents to a conic from a given point U when the conic is given by five conditions; viz., by five tangents, or by four tangents and the point of contact on one of them, or by three tangents and the points of contact on two of them.

On the left, consider the sheaves of rays which generate the conic (Art. 46) and the projectivity which they determine upon the given line u. The points in which the line meets the conic are the double points of this projectivity.

In particular, if the given line is the ideal line in the plane, the conic determined by the five conditions will be an ellipse, a para-

bola, or a hyperbola, according as the projectivity determined upon the ideal line has no double points, one double point, or two double points.

Similarly, on the right, consider the projectively related point-rows which generate the envelope of tangents to the conic and the projectivity which they establish when projected from the given point U. The tangents from U are the double rays of this projectivity.

Exercises

1. Of a projectivity on a conic, one double point and two pairs of corresponding points are known. Construct a number of pairs of corresponding points, in particular, the other double point.

2. Of a projectivity on an envelope of second class, one double ray and two pairs of corresponding rays are known. Construct a number of pairs of corresponding rays, in particular, the other double ray.

3. A conic is given by three tangents and the points of contact on two of them; determine the points of intersection of a given line with the conic.

4. A conic is given by five tangents, no two of which are parallel. Construct a circumscribing rectangle.

5. A conic is tangent to the axes of coördinates at the points (0, 4) and (4, 0) and passes through the point (1, 1). Show by construction that the conic is a parabola.

6. Determine by construction the nature of the conic which passes through the points whose coördinates are $(-1, 4)$, $(-2, 0)$, $(0, 3)$, $(1, -2)$, and $(2, 3)$.

7. A cone is given by five of its rays. Show how to construct the points in which a given line meets the cone.

8. A ruled surface of second order is determined by three rays of regulus. Show how to construct the points in which a given line meets the surface.

99. Classification of Projectivities on a Form.—A projectivity on any form is called **hyperbolic, parabolic,** or **elliptic,** according as it has two double elements, one double element, or no double elements. Thus, a projectivity on a conic is hyperbolic, parabolic, or elliptic, according as its axis meets the conic in two distinct points, is tangent to the conic, or does not meet the conic at all.

100. Cyclic Projectivities.—Suppose in a given projectivity on any form, an element E corresponds to E_1 and, in turn, E_1 corresponds to E_2, E_2 to E_3, and so on. We thus obtain a series of elements

$$E, E_1, E_2, E_3, \text{——}, E_n, \text{——}$$

of which each corresponds to the next one following it. Either this series terminates so that E_n, say, coincides with E (n being a finite integer), or else the series never terminates and so contains an infinite number of distinct elements. In the first case, the projectivity is called **cyclic**; and in the second, **non-cyclic.**

In the case of a cyclic projectivity, the series of elements

$$E, E_1, E_2, E_3, \text{——}, E_{n-1} E_n,$$

where E_n coincides with E, is called a **cycle**; the integer n is the **order** of the cyclic projectivity.

101. Construction of Cyclic Projectivities.—In what follows we shall demonstrate the existence of cyclic projectivities by constructing those of lower orders. Since a projectivity on any form can be reduced to a projectivity on a conic (Art. 97), we may confine the attention to projectivities on a conic.

1. *Cyclic Projectivities of Order* 2.—Let A and B (Fig. 75) form a cycle; and let any other point, as C, correspond to D. The projectivity is then determined by the correspondence

$$ABC\text{——}$$
$$BAD\text{——}.$$

The pairs of lines BD, AC and AD, BC intersect upon the axis z. Any number of pairs of corresponding points can be immediately constructed. In particular, the point D corresponds to C, so that D and C form a second cycle of the projectivity. Again, the point E corresponds to F, and F corresponds to E; since the lines DE and CF intersect on the axis. The points E and F thus form a third cycle.

From the correspondence

$$ABCDEF\text{——}$$
$$BADCFE\text{——},$$

we see that the tangents at A and B intersect on the axis z. Also, the tangents at C and D, and the tangents at E and F, intersect on

the axis. This also follows from the properties of inscribed quadrangles (Art. 57).

Again (Art. 57), the lines AB, CD, EF pass through Z, the center of the projectivity. Hence we have the following important property:

If a projectivity on a conic is cyclic and of order 2, the lines joining corresponding points pass through the center of the projectivity; and the tangents at corresponding points meet on the axis of the projectivity.

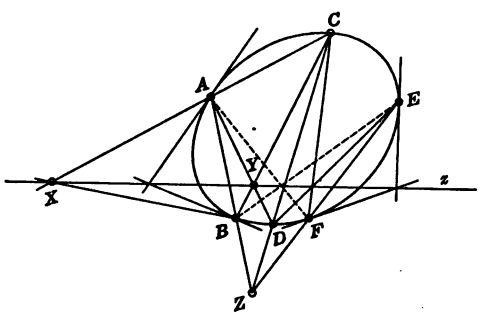


FIG. 75.

Conversely: *The rays of a sheaf of rays of first class, whose center is not on a conic, cut the conic in pairs of points of a cylic projectivity of order 2.*

For a pair of rays cuts the conic in the vertices of an inscribed quadrangle. Opposite vertices of this quadrangle are pairs of points in a cyclic projectivity of order 2. Any other ray of the sheaf cuts the conic in a pair of points belonging to the same projectivity.

It follows that four points A, B, C, D, chosen arbitrarily on a conic, determine three cyclic projectivities of order 2, namely:

$$\begin{array}{lll} ABCD\text{——} & ABCD\text{——} & ABCD\text{——} \\ BADC\text{——}; & CDAB\text{——}; & DCBA\text{——} \end{array}$$

and consequently

$$ABCD \overline{\wedge} BADC \overline{\wedge} CDAB \overline{\wedge} DCBA.$$

The axes of these projectivities are the sides of the diagonal triangle

determined by the quadrangle $ABCD$ (Fig. 75), and their centers are the vertices of the diagonal triangle. Two of the projectivities so determined are hyperbolic and one is elliptic.

In conclusion we can say:

A cyclic projectivity of order 2 *on any form is completely determined by two pairs of corresponding elements.*

Also:

If 1, 2, 3, 4 *are four elements in definite order, chosen arbitrarily on any form, then*

$$1\ 2\ 3\ 4 \overline{\wedge} 2\ 1\ 4\ 3 \overline{\wedge} 3\ 4\ 1\ 2 \overline{\wedge} 4\ 3\ 2\ 1.$$

2. *Cyclic Projectivities of Order* 3.—Any three points A, B, C of a conic may be chosen for one cycle of the projectivity. The projectivity is then entirely determined by the correspondence

$$ABC——$$
$$BCA——.$$

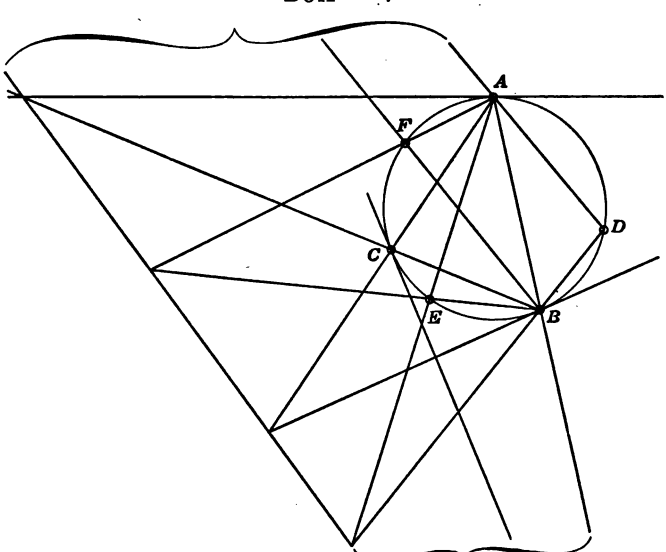


FIG. 76.

Each side of the triangle ABC meets the tangent at the opposite vertex on the axis of the projectivity (Fig. 76). The tangents at

A, B, C form a circumscribed triangle; and the lines joining a vertex of this triangle to the point of contact on the opposite side pass through the center of the projectivity (*cf.* triangle theorem, Art. 63, exercises 6 and 7).

Given any fourth point D, considered as a point of the point-row ABC——, we can immediately construct its corresponding point E of the point-row BCA——. Similarly, to E of ABC—— corresponds F of BCA——. Now to F of ABC—— must correspond D of BCA——. For, if F does not correspond to D, suppose it corresponds to D'. The correspondence is then

$$ABCDEF——$$
$$BCAEFD'——.$$

Hence the lines AD, CE, BF, and AD' intersect in the same point on the axis. But this cannot happen unless D' coincides with D. Therefore, the points D, E, F form a second cycle of the projectivity. Any number of cycles can be constructed in a similar manner.

3. *Cyclic Projectivities of Order* 4.—Four points of a conic cannot be chosen arbitrarily to form one cycle of a projectivity, since three pairs of points completely determine a projectivity. If, then, the

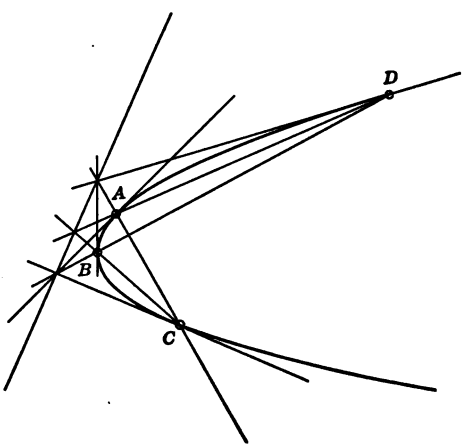


FIG. 77.

points A, B, C, D form one cycle of a projectivity, the correspondence

$$ABCD——$$
$$BCDA——$$

shows that the tangents at B and D meet the axis of the projectivity where it is met by the line AC. It follows that the diagonals of the quadrangle $ABCD$ are conjugate lines with respect to the conic (Fig. 77). Hence the points A, B, C, D must be harmonic points (Art. 88) in order to form one cycle of a projectivity.

Conversely, if $ABCD$ are any four harmonic points of a conic, it is easy to show that the projectivity determined by the correspondence

$$ABC——$$
$$BCD——$$

is a cyclic projectivity of order 4 (*cf.* Art. 22).

4. *Cyclic Projectivities of Higher Order.*—The following theorem holds for cyclic projectivities of higher order:

Every cyclic projectivity of order greater than 2 *is necessarily elliptic.*

Let

$$E, E_1, E_2, E_3,——, E_{n-1}$$

be one cycle of a cyclic projectivity of order $n>2$. No two consecutive elements of this cycle can be separated by any other pair of elements of the cycle. For, suppose E and E_1 are separated by E_2 and E_3, so that

$$E, E_2, E_1, E_3,——, E_{n-1}$$

is an arrangement of the elements of the cycle such that no two consecutive elements are separated by any other pair. But this arrangement is projectively related to the arrangement

$$E_1, E_3, E_2, E_4,——, E_{n-1}, E,$$

in which E and E_1 are not separated by E_2 and E_3. This contradicts the hypothesis, and we conclude, therefore, that no two consecutive elements of a cycle can ever be separated by any other pair of elements of the cycle (*cf.* Art. 39).

As a variable element describes continuously the cycle

$$E, E_1, E_2, \text{——}, E_{n-1}$$

its corresponding element describes continuously the corresponding arrangement

$$E_1, E_2, E_3, E_4, \text{——}, E_{n-1}, E;$$

and it is obvious that the two can never coincide. Consequently there are no double elements, and the projectivity is elliptic. Note that the argument does not apply when $n = 2$.

Exercises

1. Given three points A, B, C, of a conic, construct the fourth harmonic point.

2. Given three tangents a, b, c, to a conic, construct the fourth harmonic tangent.

3. A sheaf of rays of the first order is projectively related to a sheaf of rays of the second order. Show how to construct a chain of perspectivity connecting the two forms. If the two forms are coplanar, they generate a curve of higher order; construct a number of points of this curve.

4. A projectivity on a conic is determined by three pairs of corresponding points. Construct a number of pairs of corresponding points. If corresponding points are joined by straight lines, what can be said about these lines?

5. A projectivity on a point-row of first order is determined by three pairs of corresponding points. Construct the double points, if these exist.

6. Construct a parabolic projectivity on a conic; on a straight line.

7. Construct the projectivity on a conic of which is given one pair of corresponding points and the axis, the axis being the ideal line in the plane.

8. Construct a cyclic projectivity of order 3 on a conic, the axis being the ideal line in the plane. Is the construction possible if the conic is a parabola or a hyperbola? Why?

CHAPTER XII

THE THEORY OF INVOLUTION—IMAGINARY ELEMENTS

102. Definition of Involution.—The cyclic projectivities of order 2 have a special nomenclature and a special theory principally on account of their importance in the geometry of the eight forms.

A cyclic projectivity of order 2 on any form is called an **involution** on that form. Thus, two projectively related and superposed forms constitute an involution on their common base if the projectivity they determine is cyclic and of order 2.

When an involution exists on any form, the elements of the form are said to be **paired in involution** or to be **doubly corresponding.** The correspondence between the elements is called an **involutoric correspondence,** and the form upon which the involution exists is said to be **in involution.**

Since an involution is a projectivity, it may have two, one, or no double elements. The double elements of an involution are called **focal elements.** Thus, the double points of an involution on a point-row of first, or of second, order are the **foci** of the involution; the double rays of an involution on a sheaf of rays of first, or of second, class are the **focal rays** of the involution; and the double planes of an involution on a sheaf of planes of first, or of second, class are the **focal planes** of the involution.

An involution on any form is **hyperbolic, parabolic,** or **elliptic,** according as it has two, one, or no focal elements.

103. Fundamental Theorems.—The theorems in Art. 101, 1, are fundamental for the theory of involution. They may be restated as follows:

1. *An involution on any form is completely determined by two pairs of corresponding elements.*

2. **Theorem XIII.**—*The straight lines joining corresponding points in an involution on a conic intersect in one and the same point U, called the center of the involution; and tangents at corresponding*

points meet upon one and the same straight line u, called the axis of the involution. The center and the axis are pole and polar line with respect to the conic. The axis meets the conic in the foci of the involution. The involution is hyperbolic, parabolic, or elliptic according as the center is outside, on, or inside the conic.

Since an involution on any form can be reduced by projection or section to an involution on a conic (Art. 97), the proofs in the following articles will be confined to involutions on a conic.

104. Hyperbolic Involutions.—

1. *If an involution is hyperbolic, no pair of corresponding elements is separated by any other pair of corresponding elements; and conversely.*

For the lines joining corresponding points in an hyperbolic involution on a conic meet in the external center. Hence, no two pairs of points in the involution can separate each other. Conversely, if two pairs of points in a given involution on a conic do not separate each other, the center of the involution is outside the conic, and the involution is hyperbolic.

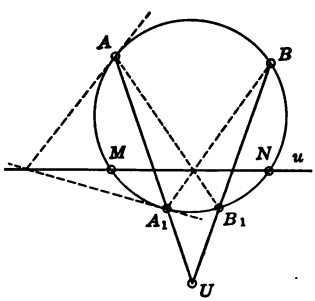


FIG. 78.

2. *Any pair of corresponding elements in a hyperbolic involution is separated harmonically by the focal elements of the involution.*

For, if A and A_1 are a pair of corresponding points in an hyperbolic involution on a conic (Fig. 78), the line AA_1 is conjugate to the axis MN with respect to the conic. The points M, A, N, A_1 are, therefore, harmonic (Art. 88). Similarly, M, B, N, B_1 are four harmonic points.

105. Elliptic Involutions.—

1. *If an involution is elliptic, any pair of corresponding elements is separated by every other pair of corresponding elements; and conversely.*

For the lines joining corresponding points of an elliptic involution on a conic meet in the internal center. Consequently every pair of points in the involution is separated by every other

pair. Conversely, if two pairs of points in a given involution on a conic separate each other, the center of the involution is inside the conic, and the involution is elliptic.

2. *Given any pair of elements in an elliptic involution, there exists one, and but one, pair of elements in the involution which separates harmonically the given pair.*

Suppose A, A_1 is any pair of points in an elliptic involution on a conic (Fig. 79). The line joining the pole of AA_1 to the center of the involution cuts the conic in the pair B, B_1. A, B, A_1, B_1 are four harmonic points (Art. 88). It is evident that the line PU must cut the conic, and that there is but one such line.

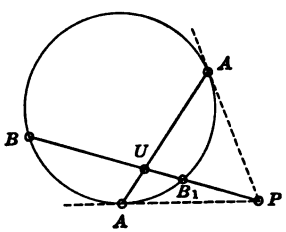


FIG. 79.

106. Parabolic Involutions.—Since the center U of a parabolic involution on a conic is itself a point of the conic, it follows that U forms one point of each pair in the involution. It will be seen, then, that a parabolic involution is not, in the true sense of the definition, a projectivity between two superposed forms. For this reason, parabolic involutions are often called trivial or degenerate.

Exercises

1. With the aid of Art. 104, 2, show that the foci of a hyperbolic involution on a straight line separate harmonically any pair of points in the involution.

2. Prove the proposition in the preceding exercise without the aid of Art. 104, 2.

Suggestions.—In Fig. 80, let M and N be the foci and A, A_1, any pair of corresponding points. Then:

(*a*) $ANA_1M \overline{\wedge} A_1NAM$; Why?
(*b*) $S(ANA_1M) \overline{\overline{\wedge}} ROPM \overline{\overline{\wedge}} A_1NAM$; Why?
(*c*) RA_1, ON, PA meet in a point Q;
(*d*) ANA_1M is a harmonic range. Why?

3. An involution on an envelope of second class is cut by any ray of the envelope in an involution on that ray. Conversely, if an involution is chosen arbitrarily upon any ray of an envelope, the rays of the

envelope passing through pairs of points in the given involution are themselves paired in involution. Hence, if an involution is chosen upon a tangent to a conic, show that tangents drawn from pairs of corresponding points intersect upon a fixed straight line.

4. The rays of a sheaf of rays of first class are paired so that the rays of any pair are mutually perpendicular. Show that the sheaf is in involution.

5. The vertices of all right angles whose sides are tangent to a given parabola lie upon a fixed straight line, and the lines joining the points of contact on a pair of sides pass through a fixed point.

6. The axis of a given involution on a conic is a diameter of the conic. Show that the tangent at either extremity of the diameter is cut by tangents at corresponding points of the involution in an involution having one focus at infinity. Where is the other focus?

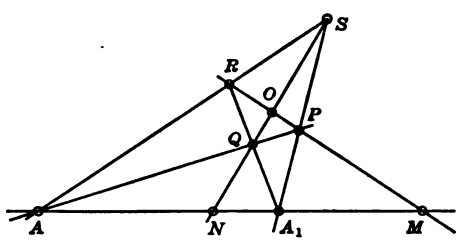


FIG. 80.

107. Involutions on a Straight Line.—

1. *If O is the mid-point between the foci M and N of a hyperbolic involution on a straight line, and A, A_1 is any pair of points in the involution; then*

$$OA \cdot OA_1 = \overline{OM}^2 = \overline{ON}^2.$$

For $MANA_1$ is a harmonic range of points. Hence Art. 30 applies. The point O corresponds to the ideal point in the involution.

Again (Fig. 81), the circles on AA_1, BB_1, CC_1, etc., as diameters cut the circle on MN orthogonally (Art. 32).

2. *An elliptic involution on a straight line can be projected from two points of any plane through the line in sheaves of rays in involution such that the rays of any pair are mutually at right angles.*

Draw circles on AA_1 and BB_1 as diameters (Fig. 82). These

circles must intersect in two points P and Q, since the involution is elliptic. Project the involution from P and thus obtain a sheaf of rays in involution in which the corresponding rays PA and PA_1, and also the rays PB and PB_1, are perpendicular to each other.

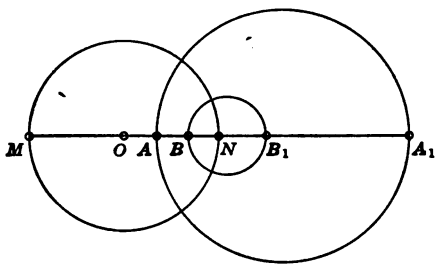


FIG. 81.

This sheaf is cut by either of the two circles in an involution whose center coincides with the center of the circle. For example, the circle PBB_1 cuts the sheaf in an involution whose center is R. It follows that the rays of any pair, in the involution on the sheaf

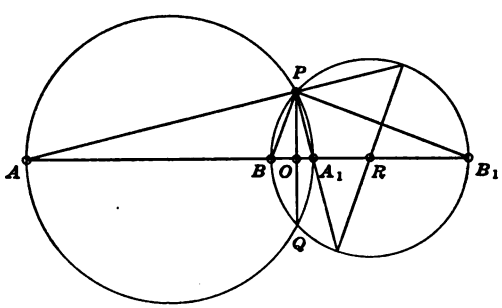


FIG. 82.

whose center is P, are perpendicular to each other. The same statements apply to the point Q. The points P and Q are, then, the two points from which the involution on the line can be projected as stated in the theorem.

As a direct consequence of this theorem, we see that the circles

constructed on AA_1, BB_1, CC_1, etc., as diameters, all pass through the points P and Q.

Again, if PQ meets AA_1 in O, then OP is a mean proportional between the segments in which O divides the diameter of any one of the circles. Since these segments are drawn in opposite directions from O, we have

$$OA \cdot OA_1 = -\overline{OP}^2,$$

where A, A_1 is any pair of points in the given involution. The point O corresponds to the ideal point in the given involution.

108. Involutions on a Sheaf of Rays of First Class.—*If the rays of a sheaf of rays of first class are paired in involution, then there is at least one pair of corresponding rays such that each ray is perpendicular to the other. If there is more than one such pair, then the rays of every pair are perpendicular to each other.*

Consider the involution determined upon a circle in perspective position with the given sheaf. That diameter of the circle which passes through the center of this involution cuts the circle in two points that are projected from the center of the sheaf in a pair of corresponding rays such that each ray is perpendicular to the other. Obviously there is one, and only one, such pair unless the center of the involution on the circle coincides with the center of the circle. But this will be the case if there are two pairs of mutually perpendicular rays, for then each pair will cut the circle at opposite ends of a diameter.

An involution on a sheaf of rays of first class such that the rays of every pair are at right angles to each other is called a **circular** involution. It follows from Art. 107, 2, that an elliptic involution on any form can be reduced by projection and section to a circular involution.

Exercises

1. In a given involution on a straight line, O corresponds to the ideal point and the corresponding points A and A_1 are situated respectively two units and eight units to the right of O. Construct a number of pairs of corresponding points, in particular, the foci.

2. If, in the preceding exercise, A and A_1 are situated respectively two units to the right and eight units to the left of O, construct a

number of pairs of corresponding points. Locate the points from which the given involution can be projected in a circular involution.

3. Given the foci of a hyperbolic involution on a straight line, construct a number of pairs of corresponding points.

4. Given the foci of a hyperbolic involution on a conic, construct a number of pairs of corresponding points.

5. Given an involution on a sheaf of rays of first class, construct the pair whose rays are mutually perpendicular.

6. Two projectively related sheaves of rays being given, how can they be brought into such a position as to form an involution?

7. If a right-angled triangle inscribed in a conic varies so that its vertex remains fixed, its hypothenuse will constantly pass through a fixed point.

8. Two fixed points A and B are chosen on a fixed tangent to a conic, show that tangents to the conic from points harmonically separating A and B intersect upon a fixed straight line.

109. Involutions Determined by a Complete Quadrangle, or a Complete Quadrilateral.—

Opposite sides of a complete quadrangle are cut by a line in pairs of points of an involution.

Opposite vertices of a complete quadrilateral are projected from a point in pairs of rays of an involution.

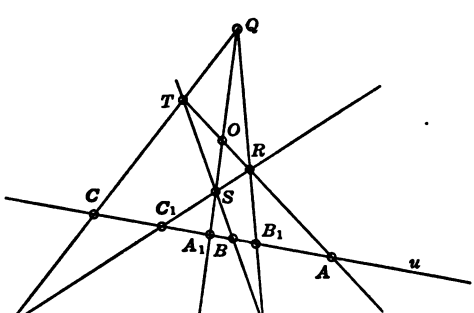


FIG. 83.

For the theorem on the left, let u (Fig. 83) cut the sides RT and SQ, ST and QR, QT and RS, in the points A and A_1, B and B_1, C and C_1, respectively, and let O be the point of intersection of QS

and RT. Project the range $ATOR$, first from Q and then from S and cut the resulting pencils of rays by the line u. We thus obtain,

$$ACA_1B_1 \overline{\overline{\wedge}} ATOR \overline{\overline{\wedge}} ABA_1C_1.$$

But $ABA_1C_1 \overline{\wedge} A_1C_1AB$ (Art. 101, 1). Hence,

$$ACA_1B_1 \overline{\wedge} A_1C_1AB,$$

and consequently the involution determined by the doubly corresponding pair A, A_1 and the pair B, B_1 will contain the pair C, C_1. This proves the theorem.

Exercises

1. Prove the theorem on the right by means of the principle of duality. Construct the figure.

2. In the theorem on the left, show that, if u cuts the quadrangle so that two or four of the vertices are on the same side of u the involution determined upon u is hyperbolic; if u cuts the quadrangle so that one vertex is on one side and three vertices on the other, the involution is elliptic; and if u passes through a vertex of the quadrangle the involution is parabolic.

3. In the theorem on the right, the sides of the quadrilateral divide the plane into regions. In which regions must a point be placed so that the involution determined by the quadrilateral shall be hyperbolic? Elliptic? Parabolic?

4. Given two pairs of points, A, A_1 and B, B_1, in an involution on a straight line. Show how to construct the point corresponding to C in the involution by means of a complete quadrangle.

110. Desargues Theorem and Its Dual.—

The points of intersection of a fixed straight line with conics circumscribed about a complete quadrangle form pairs of points in the involution determined upon the line by the quadrangle.

The tangents drawn from a fixed point to conics inscribed in a complete quadrilateral form pairs of rays in the involution determined about the point by the quadrilateral.

For the theorem on the left, let $QRST$ be the complete quadrangle (Fig. 84), and u the fixed straight line meeting the sides TQ, QR, RS, and ST in the points A, B, A_1, and B_1, respectively. Let any conic circumscribed about the quadrangle meet u in the points P and P_1. The pencils $S(TPRP_1)$ and $Q(TPRP_1)$ are projectively related since corresponding rays meet on the conic. Hence,

$B_1PA_1P_1 \overline{\wedge} APBP_1 \overline{\wedge} BP_1AP$ (Art. 101, 1);

and therefore P and P_1 are doubly corresponding points in the involution in which A, A_1 and B, B_1 are two pairs of corresponding points. But this involution is determined by the complete quadrangle $QRST$.

Exercises

1. Draw a figure and prove the theorem on the right.

2. A hyperbolic involution on a straight line is determined by a complete quadrangle, construct the foci of the involution.

3. A hyperbolic involution about a point is determined by a complete quadrilateral, construct the focal rays of the involution.

4. In general, two conics can be drawn through four given points and tangent to a given line. Construct the points of contact with the given line.

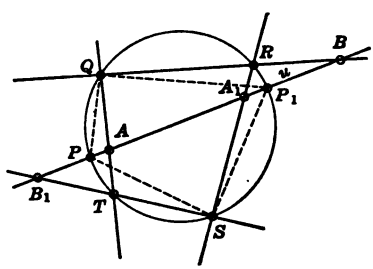


FIG. 84.

5. How many parabolas can be drawn through four given points in a plane? Under what circumstances are there no parabolas through four given points?

6. Construct the axes of the parabolas that can be drawn through four given points.

7. In general, two conics can be drawn to touch four given lines and pass through a given point. Construct the tangents to the conics at the given point.

8. How many parabolas can be drawn to touch the sides of a given triangle and pass through a given point? Construct the points of contact of the parobolas that touch the sides of the triangle ABC and pass through the point P.

111. Involutions Determined by a Fixed Conic.—

A fixed conic determines upon any straight line in its plane an involution such that the points of each pair are conjugate with respect to the conic.

A fixed conic determines about any point in its plane an involution, such that the rays of each pair are conjugate with respect to the conic.

Any straight line in the plane of a fixed conic is the support of two point-rows so related that, with respect to the conic, any point of either is conjugate to some point of the other. Either point-row is a section of the sheaf of polar lines corresponding to the other. The two point-rows are therefore projectively related (Art. 69, 1) and thus constitute a projectivity upon their common base. Since corresponding points are conjugate with respect to the conic, the projectivity is cyclic and of order 2. It is, therefore, an involution.

Similarly, any point in the plane of a fixed conic is the support of two projectively related sheaves of rays, corresponding rays being conjugate with respect to the conic. The two sheaves thus constitute an involution about their common center.

The foci of the involution determined upon any line by a fixed conic are the points in which the conic meets the line.

For the points common to the line and the conic are self-conjugate with respect to the conic.

The involution is, therefore, hyperbolic, parabolic, or elliptic according as the line meets the conic in two distinct points, is tangent to the conic, or does not meet the conic at all.

The focal rays of the involution determined about any point by a fixed conic are the tangents which can be drawn from the point to the conic.

For these tangents are self-conjugate with respect to the conic. The involution is therefore hyperbolic, parabolic, or elliptic according as the point is outside, on, or inside, the conic. In particular, if the point is the center of the conic, the involution consists of pairs of conjugate diameters. If the conic is a hyperbola, the focal rays are the asymptotes. Since there is always at least one pair of rays at right angles in every involution on a sheaf of rays (Art. 108), we see that there must always be at least one pair of conjugate diameters at right angles (*cf.* Art. 76). If there is more than one such pair, the conic is necessarily a circle.

Exercises

1. If an involution on a conic is projected from any point of the conic upon the axis of the involution, show that the resulting involution on the axis consists of pairs of points conjugate with respect to the conic (*cf.* Art. 69, 3).

2. Write out and prove the dual of exercise 1.

3. Given two pairs of points; A, A_1 and B, B_1; of an involution on a straight line. By means of a complete quadrangle, construct the point C_1 corresponding to any fifth point C; and in particular, the point corresponding to the ideal point of the line.

4. Two pairs of conjugate diameters of a conic are known, draw the conjugate to any fifth diameter and construct the axes.

5. If two pairs of opposite sides of a complete quadrangle are at right angles, show that the third pair is also at right angles.

6. By the preceding exercise, show that the perpendiculars let fall from the vertices of a triangle upon the sides opposite meet in a point. This point is called the *orthocenter* of the triangle.

7. Prove that all conics which pass through the vertices and the orthocenter of a triangle are equilateral hyperbolas.

8. Show that an equilateral hyperbola can always be circumscribed about any quadrangle. This hyperbola passes through the orthocenters of the four triangles formed by the vertices of the quadrangle.

9. The sides of any triangle form with the ideal line in the plane a complete quadrilateral whose three pairs of opposite vertices are projected from the orthocenter of the triangle in three pairs of rays at right angles.

10. By the preceding exercise, show that the two tangents which can be drawn from the orthocenter of any triangle circumscribed about a parabola are at right angles to each other; and hence the orthocenters of all triangles circumscribed about a parabola lie on a fixed straight line.

11. How does the preceding exercise show that the orthocenters of the four triangles formed by the sides of any complete quadrilateral lie on one straight line?

12. If a circle is circumscribed about the rectangle $ABCD$, then the tangents drawn from any point S of this circle to any conic inscribed in the rectangle are at right angles to each other.

13. The vertices of all right angles whose sides touch an ellipse or a hyperbola lie upon a circle.

14. If any two of the three circles which have the diagonals of a complete quadrilateral for diameters intersect; then the third circle passes through the points of intersection.

15. If a quadrangle is inscribed in a circle, show that all hyperbolas passing through the vertices of the quadrangle have parallel axes.

112. Imaginary Points on a Straight Line.—A hyperbolic involution on a straight line defines two real points; namely, the foci

of the involution. For the foci can be constructed as soon as we know two pairs of points of the involution.

Since the involution is determined by any two pairs of corresponding points, say A, A_1 and B, B_1, we may distinguish between the foci by the following convention: reading the points in an order such that the pair B, B_1 lies to the right of the pair A, A_1 shall define that focus which separates B, B_1 from either A or A_1; and reading the points in reverse order, so that the pair A, A_1 lies to the right of B, B_1, shall define the focus that separates the pair A, A_1 from either B or B_1. Hence we can say:

Two pairs of points A, A_1 and B, B_1, lying on the same straight line, but not separating each other, together with the direction in which they are read, serve to define absolutely a real point of the line which is one focus of the hyperbolic involution determined by the given pairs.

In an entirely analogous manner, we shall say that an elliptic involution on a straight line defines two **conjugate imaginary** points which may be thought of as the foci of the involution.

An elliptic involution is determined by two pairs of points A, A_1 and B, B_1, which separate each other; and we shall distinguish between the imaginary points which the involution defines by the direction in which the given pairs are read. Thus, ABA_1B_1 defines one imaginary point while B_1A_1BA defines the conjugate imaginary point. Hence:

Two pairs of points A, A_1 and B, B_1, lying on the same straight lines, and separating each other, together with the direction in which they are read, define an imaginary point. The conjugate imaginary point is defined by reading the given pairs in reverse order.

As an immediate consequence of this definition:

Two conjugate imaginary points are joined by the real line on which lies the elliptic involution defining them.

113. Imaginary Lines in a Plane.—An elliptic involution on a sheaf of rays of first class defines two conjugate imaginary lines in the plane of the sheaf. Since an elliptic involution on a sheaf of rays of first class is determined by two pairs of rays which separate each other we have the definition:

Two pairs of rays, a, a_1 and b, b_1 of a sheaf of rays of first class which separate each other, together with the order in which they are

taken, define an imaginary line. The conjugate imaginary line is defined by taking the given pairs in reverse order.

As a consequence of this definition:

Two conjugate imaginary lines in the same plane always intersect in a real point. This point is the support of the sheaf of rays of first class upon which lies the elliptic involution defining the imaginary lines.

114. Imaginary Planes.—An elliptic involution on a sheaf of planes of first class defines two conjugate imaginary planes. These imaginary planes intersect in the axis of the sheaf of planes. If the involution is determined by the two pairs of planes α, α_1 and β, β_1, then we shall say that the order $\alpha\beta\alpha_1\beta_1$ defines one of the imaginary planes while the reverse order $\beta_1\alpha_1\beta\alpha$ defines the conjugate imaginary plane.

115. Construction Problems.

1. *To construct the line joining a real point to an imaginary point.*

To construct the point of intersection of a real line with an imaginary line.

Let S be the real point and let ABA_1B_1 be the imaginary point (Fig. 85). Project ABA_1B_1 from S and thus obtain the imaginary line aba_1b_1. This is the line to be constructed. The conjugate imaginary line b_1a_1ba joins the conjugate imaginary point B_1A_1BA to S.

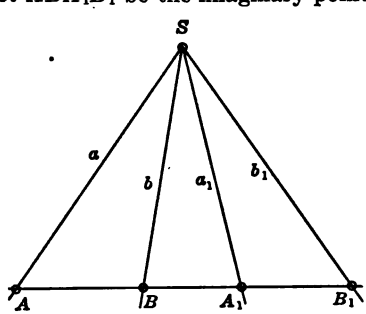


FIG. 85.

If u is a real line and aba_1b_1 an imaginary line lying in the same plane, they intersect in the imaginary point ABA_1B_1. The conjugate imaginary line intersects u in the conjugate imaginary point B_1A_1BA.

If we project an imaginary line from a real point not lying in the plane of its defining sheaf, we obtain an imaginary plane; and if we cut an imaginary plane by a real plane not passing through the axis of its defining sheaf, we obtain an imaginary line.

2. *To construct the line joining two imaginary points in a plane.*

Let two imaginary points be defined by elliptic involutions along the lines u and u' (Fig. 86). Let A be the point common to u and u' and let A_1 correspond to A in the involution on u, and A'_1 correspond to A in the involution on u'. Construct the pair B, B_1 belonging to the involution and separating the pair A, A_1 harmonically (Art. 105, 2). Similarly, construct the pair B', B'_1 separating

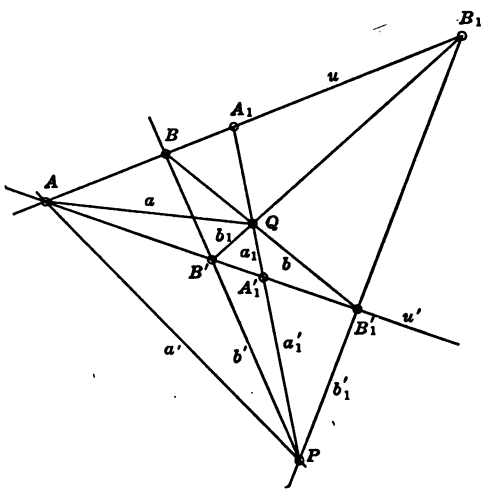


FIG. 86.

A, A'_1 harmonically. The harmonic ranges ABA_1B_1 and $AB'A'_1B'_1$ are sections of one and the same pencil $a'b'a'_1b'_1$. The elliptic involution determined by the pairs of rays a', a'_1 and b^1, $b^1{}_1$, defines the imaginary line joining the given imaginary points. The conjugate imaginary points are joined by the conjugate imaginary line.

The imaginary point on one line can be joined to the conjugate imaginary point on the other. Thus, the harmonic ranges ABA_1B_1 and $AB'_1A'_1B'$ are sections of the pencil aba_1b_1 whose center is Q. The pairs a, a_1 and b, b_1 define the line sought.

Similarly, if two imaginary lines are defined by elliptic involutions on sheaves of rays whose centers are P and Q, we can select harmonically separated pairs of rays and so obtain harmonic pen-

cils in perspective position as in the figure. When the sheaves are taken in the same order, they define lines intersecting in the imaginary point ABA_1B_1 or its conjugate imaginary point. When the sheaves are taken in opposite orders, they define lines intersecting in the imaginary point $AB'A'_1B'_1$ or in its conjugate imaginary point.

116. Imaginary Elements on Any Form.—An elliptic involution on any form defines a pair of conjugate imaginary elements on that form. Thus, an elliptic involution on a conic defines a pair of conjugate imaginary points on the conic. These points are joined by the axis of the involution (Art. 111, exercise 1).

Reciprocally, an elliptic involution on an envelope defines a pair of conjugate imaginary lines belonging to the envelope. These lines intersect in the center of the involution.

The introduction and definition of imaginary elements enables us to say:

A conic meets every real line in its plane in two points which are real and distinct, coincident, or conjugate imaginary, according as the involution determined upon the line by the conic is hyperbolic, parabolic, or elliptic.

A conic has two tangents from every real point in its plane which are real and distinct, coincident, or conjugate imaginary, according as the involution determined about the point by the conic is hyperbolic, parabolic, or elliptic.

By projection, these statements are at once carried over to corresponding statements concerning the cone and the sheaf of planes of second class.

An elliptic involution on a regulus defines a pair of conjugate imaginary lines which do not lie in any real plane nor pass through any real point.

Conjugate imaginary lines defined by an elliptic involution on a regulus are often called conjugate imaginary lines of the **second kind.** Conjugate imaginary lines of the **first kind** always lie in a real plane and intersect in a real point.

117. Construction Problems.—

1. *To construct the conic passing through five points of a plane, three of the given points being real and two being conjugate imaginary points.*

Let P, Q, and R be the real points, and let the conjugate imaginary points be defined by an elliptic involution on the line u (Fig. 87). Let PQ meet u in A, and QR meet u in B. To A corresponds a point A_1 in the involution, and to B corresponds a point B_1. The pairs A, A_1 and B, B_1 must necessarily separate each other, since the involution is elliptic. These pairs are also conjugate pairs of points with respect to the conic which we are to construct. Consequently, the polar lines of A and B pass through A_1 and B_1 respectively. But the polar line of A passes through S, the harmonic conjugate of A with respect to P and Q, and the

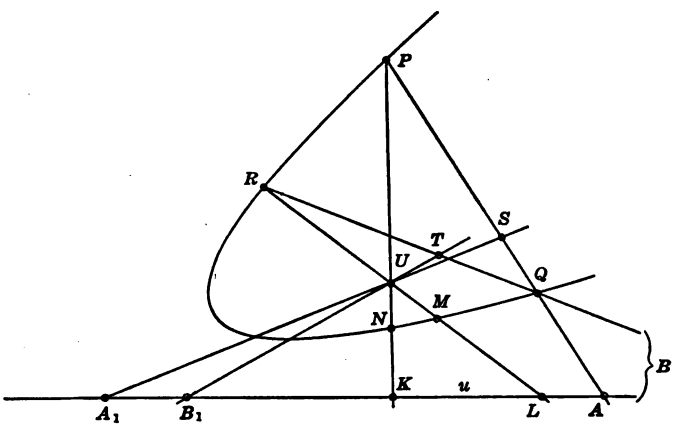


FIG. 87.

polar line of B passes through T, the harmonic conjugate of B with respect to R and Q. Therefore these polar lines can be constructed. They intersect in U, the pole of u with respect to the conic. Let the line RU meet u in L, and the line PU meet u in K. The conic must pass through M and N, the harmonic conjugates of R and P with respect to the pairs U, L and U, K, respectively. We thus have five real points, P, Q, R, M, N, through which the conic must pass. Other points of the conic can now be constructed by previous methods.

2. *To construct the conic passing through one real point and two pairs of conjugate imaginary points, no three of the given points lying on the same straight line.*

Let P be the real point, and let the pairs of conjugate imaginary points be defined by elliptic involutions on the lines u and u' (Fig. 88). We may suppose that these involutions are determined by pairs of points which separate each other harmonically (Art. 105, 2). Let these pairs be A, A_1 and B, B_1 on u, and A, A'_1 and B', B'_1 on u', where A is the point common to u and u'. Since corresponding points in the involutions on u and u' are conjugate with respect to the conic we are to construct, the polar line of A must pass through both A_1 and A'_1. If the line PA meets the

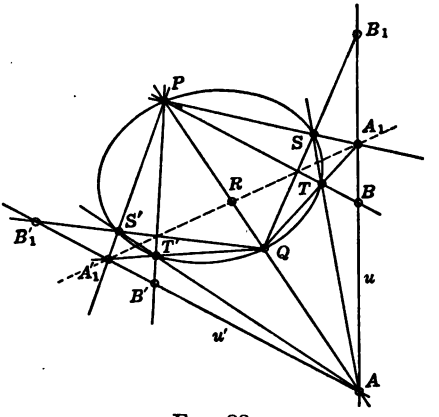


FIG. 88.

polar line of A in R, then the harmonic conjugate of P with respect to A and R must lie on the curve. Let Q be this point. We can now construct the quadrangle $PQST$ whose pairs of opposite sides intersect in A and A_1 and whose diagonals pass through B and B_1. This quadrangle is inscribed in the conic. Similarly, we can construct the quadrangle $PQS'T'$ which is also inscribed in the conic. We thus have six real points on the conic, and as many more can be constructed as may be desired by previously given methods.

3. *Given two involutions lying upon the same form; to construct the pair of elements belonging to both involutions.*

Suppose the two involutions are upon the same conic, and let U and V be the centers of the involutions (Fig. 89). The line UV

cuts from the conic the pair of points required, since these points obviously correspond to each other in both involutions.

The pair of points common to two given involutions on the same conic will be real if the involutions are both elliptic, or if one is elliptic and the other is hyperbolic, as in the figure. For, in these cases, the line UV must cut the conic in real points.

If both the given involutions are hyperbolic, their common pair of points will be real or conjugate imaginary, according as the line joining their centers cuts the conic in real points or in conjugate imaginary points.

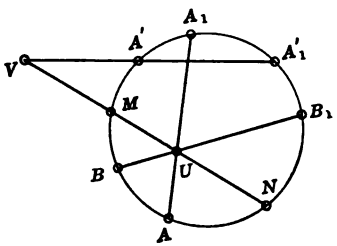


FIG. 89.

In any case, the pair of points common to two given involutions on a conic is defined by the involution which the conic determines upon the line joining their centers.

4. *Given an involution on a straight line; to construct the pair of points belonging to the involution which separates harmonically any two real points on the line.*

Let M and N be the two real points on the line. Project the given involution and the points M and N from a point S not on the line. We thus have an involution about S. Consider the rays SM and SN as the focal rays of a second involution about S. When we cut the two involutions about S by a conic passing through S, we obtain two involutions on the conic. The pair of points common to these involutions project from S in rays which are cut by the given line in the pair of points required.

Exercises

1. Construct the conic tangent to five lines in the plane two of which are conjugate imaginary lines.

2. Two involutions lie on the same straight line, construct the pair of points belonging to both involutions.

3. Given an involution on a sheaf of rays of first class; construct the pair of rays belonging to the involution and separating harmonically any given pair of rays.

4. Given an involution on a straight line; construct that pair of points which belongs to the involution and is bisected by a given point.

5. Through a given point draw a pair of lines which will intercept equal segments on two given lines. Is there more than one such pair?

6. In a given conic inscribe a triangle whose sides shall pass through three given points.

Suggestion.—Consider the three given points as the centers of three involutions on the conic. Let A correspond to A_1 in the first involution, A_1 to A_2 in the second, and A_2 to A_3 in the third. As A describes the conic, A_3 will describe a projectively related conic. Consider the double points of the projectivity so determined. Discuss the problem for various positions of the given points. Can the problem be generalized for any polygon?

7. State the dual of the preceding exercise and show how the construction can be made.

8. Show that every circle in a plane passes through the same conjugate imaginary points on the ideal line.

9. Given a point-row u and a sheaf of rays U projectively related to each other; find a point from which u can be projected in a sheaf equiangular with U.

NOTE.—The student will scarcely fail to notice the difference in character between the problems in Chapter XI and XII and the problems in previous chapters. The earlier problems, with two exceptions, are problems of the **first degree**; that is, problems admitting of but one solution. Such problems are reducible to the problem of finding an element in one of two projectively related forms, corresponding to a given element in the other; and are solved by means of a chain of perspectivity connecting the two forms. The necessary constructions are carried out with the aid of a straight edge, or ruler. The exceptions noted above are the problem of finding a pair of points harmonically separating two given pairs (Art. 32, exercise 7) and the problem of constructing the axes of an ellipse or of a hyperbola (Art. 76). Each of these problems has two solutions and each requires the use of a circle. The solution of a problem of the first degree is comparable to the solution of a linear equation.

Problems of the **second degree** admit of at most two solutions. Such problems are reducible to the problem of constructing the double points of a projectivity on a conic, or the foci of an involution on a conic. The solution of a problem of the second degree is equivalent to the solution of a quadratic equation.

The introduction of imaginary elements into projective geometry is due to Von Staudt (1847).

CHAPTER XIII

THE FOCI AND FOCAL PROPERTIES OF CONICS

118. Definition of Focus.—We have seen (Art. 111) that a conic sets up about any point in its plane an involution such that the rays of any pair are conjugate lines with respect to the conic. This involution has at least one pair of rays at right angles (Art. 108). These rays are called **normal conjugate rays.** If the involution has more than one pair of normal conjugate rays, then the rays of every pair are perpendicular to each other; and the involution is circular (Art. 108).

If an involution which a conic determines about a point F is circular, then F is a **focus** *of the conic.*

It follows from this definition that a focus is necessarily inside a conic. For the involution set up about a point outside the conic, or on the conic, is certainly not circular.

Again, a focus must be on an axis of the conic. For, if F is a focus and C is the center, the diameter FC is not perpendicular to its conjugate chord through F, unless FC is an axis.

If the center of a conic is a focus, then every diameter is perpendicular to its conjugate diameter, and the conic is necessarily a circle.

Conversely, the center of a circle is a focus. For every diameter of a circle is perpendicular to its conjugate diameter. It is evident that the circle can have but this one focus, since the involution determined about any other point has but one pair of normal conjugate rays.

119. Construction of Foci.—

1. *The Ellipse.*—Let AB and CD be the axes of an ellipse (Fig. 90). Draw the tangents at the extremities of these axes, thus circumscribing a rectangle $KLMN$ about the ellipse. The segments, AB and CD, which the ellipse determines upon its axes, are unequal in length. For, if these segments were equal in length,

then $KLMN$ would be a square and we could inscribe a circle in it which would have four points and the tangents at them in common with the ellipse. The ellipse would then coincide throughout with the circle. Hence, the segments AB and CD are unequal in length. The longer of the two segments is the **major axis** of the ellipse, and the shorter is the minor axis, or **conjugate axis.**

Suppose AB, the major axis, is parallel to the sides LM, KN of the circumscribed rectangle. The circle drawn with LM as a diameter cuts AB in two real points, F and F_1, which are foci of the ellipse. For the lines KL, LM, and MN are the sides of a triangle circumscribed about the ellipse, one vertex being at infinity. The points F and F_1 are conjugate to this ideal vertex with respect to the ellipse. Consequently, the lines LF and MF are conjugate with respect to the ellipse (Art. 69, 3), and thus form one pair of normal conjugate rays in the involution set up about F. The axis AB and its conjugate chord through F form a second pair of normal conjugate rays in this involution. Consequently, the involution about F is circular and F is a focus. Similarly, F_1 is a focus.

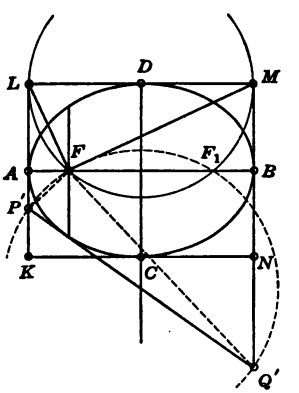


FIG. 90.

The segment of any tangent, contained between the tangents at A and B, subtends a right angle at F or at F_1. Thus, $P'FQ'$ is a right angle, since $P'F$ and $Q'F$ are conjugate lines with respect to the ellipse and consequently form a pair of rays in the involution set up about F by the ellipse. Hence, the circle on $P'Q'$ as a diameter passes through F. Similarly, this circle also passes through F_1. In this way is obtained a system of circles each of which passes through the foci and cuts the tangents at A and B at opposite ends of diameters.

It is clear, from the foregoing discussion, that F and F_1 are the only foci on the major axis of the ellipse. Since the circle

on KL as a diameter cannot cut the minor axis in real points, it follows that there are no real foci on the minor axis. Consequently, F and F_1 are the only real foci of the ellipse.

2. *The Hyperbola.*—Only one of the axes of a hyperbola meets the curve in real points. This is the major axis AB (Fig. 91). Draw tangents to the curve at A and B meeting the asymptotes in the points K, M and L, N, respectively. The circle circumscribed about the rectangle $KMNL$ cuts AB in the foci F and F_1. For the lines KM, ML, and LN form the sides of

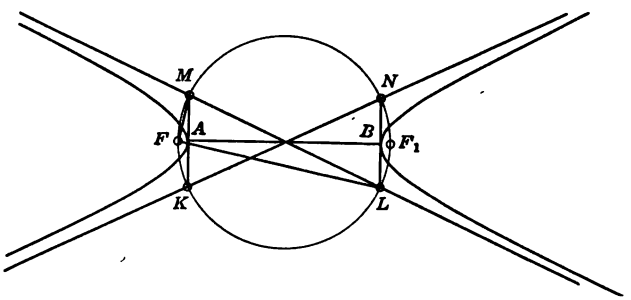


FIG. 91.

a triangle circumscribed about the hyperbola, one vertex being infinitely distant. The points F and F_1 are conjugate to the ideal vertex and are, therefore, joined to M and L by conjugate lines. But MF and LF are at right angles to each other, and the axis is perpendicular to its conjugate chord through F. Hence, F is a focus. Similarly, F_1 is a focus. It follows, as in the case of the ellipse, that F and F_1 are the only real foci.

3. *The Parabola.*—Let S be the vertex of the parabola (Fig. 92). Draw the axis and the tangent at S. Also, a second tangent, meeting the first at N, whose point of contact is A. The lines SN and NA, together with the ideal line, form a circumscribed triangle two of whose vertices are ideal points. Any point on the axis, as F, is conjugate to the ideal vertex on SN and is joined to the other two vertices by conjugate lines. But the line joining F to the ideal vertex on NA is parallel to NA. If, then, NF is drawn perpendicular to NA, its conjugate through F is perpendicular to it, and the two constitute a pair of normal con-

jugate rays in the involution set up about F by the parabola. The axis and its conjugate chord through F form a second pair of normal conjugate rays in the same involution. Hence, F is a focus.

The parabola has but one proper focus. It is sometimes convenient to regard the ideal point on the axis as a secondary, or improper, focus.

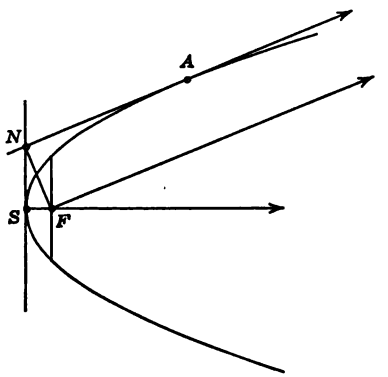


FIG. 92.

Exercises

1. Show that the circle on MN as diameter (Fig. 90) cuts AB in points harmonically separated by F, F_1.

2. Given the major axis AB and one focus of an ellipse, or of a hyperbola, construct a series of tangents to the curve.

3. If a and b are respectively the lengths of the semi-major axis and the semi-minor axis of an ellipse, show that the distance between the foci is $2\sqrt{a^2 - b^2}$.

4. If $2a$ is the length of the major axis of a hyperbola and $2b$ is the length of the segment of a tangent at one vertex contained between the asymptotes, show that the distance between the foci is $2\sqrt{a^2 + b^2}$.

5. Given the vertex and focus of a parabola, construct a series of tangents to the curve.

120. Directrices and Focal Radii.—The polar line of a focus with respect to a conic is called a **directrix.**

A circle has but one directrix, the ideal line in its plane.

A parabola has but one proper directrix. This directrix cuts the axis in the harmonic conjugate of the focus with respect to the two curve points on the axis. In other words:

A parabola bisects the segment of its axis contained between the directrix and the focus.

An ellipse, or a hyperbola, has two directrices neither of which meets the curve in real points.

Lines joining any point on a conic to the foci are called **focal radii,** or **focal rays,** of the conic.

From each point of a circle there is but one focal radius, namely, the radius of the circle drawn to that point.

Likewise, from each point of a parabola there is but one proper focal radius; but it is often convenient to consider the diameter through the point as a second focal radius, that is, as the radius drawn to the ideal, or improper, focus.

There are two focal radii from each point of an ellipse, or a hyperbola.

FOCAL PROPERTIES OF CONICS

121. Fundamental Theorem.—Theorem XIV.—*The line joining a focus, F, of a conic to the intersection of any two tangents to the conic bisects one of the angles between the focal radii from F to the points of contact of the tangents.*

Let A and B be the points of contact of tangents drawn from T, and let f be the directrix corresponding to the focus F (Fig. 93). If AB meets the directrix in P, then the polar line of P is TF. Hence TF and PF are conjugate lines with respect to the conic and consequently perpendicular to each other (definition of focus). Since the points A and B are separated harmonically by P and its polar line TF, we have a harmonic pencil of rays about F in which TF and PF are conjugate to each other and hence bisect the interior and exterior angles between the other two (Art. 27). This proves the theorem.

122. Consequences of Theorem XIV.—

1. *The finite segment of a variable tangent to a conic, contained between two fixed tangents, subtends a constant angle at a focus.*

For, if TA and TB are fixed tangents with points of contact at A and B, respectively, and A_1B_1 is the segment of a variable tan-

gent contained between the fixed tangents, and touching the conic at C, then A_1F bisects the angle CFA, and B_1F bisects the angle

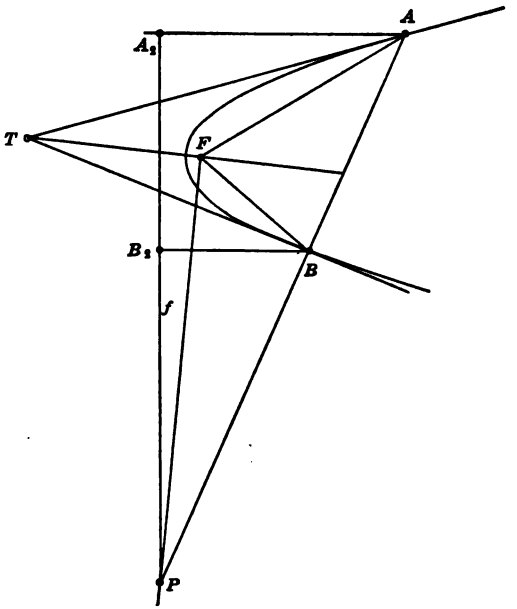


FIG. 93.

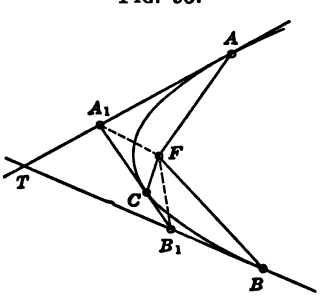


FIG. 94.

CFB. Consequently, the angle A_1FB_1 is half the angle AFB for all positions of the variable tangent A_1B_1. But AFB is a fixed angle (Fig. 94).

As a direct consequence of the property just proved, we have:

The projectively related point-rows in which any two tangents to a conic are cut by the remaining tangents are projected from a focus of the conic in two projectively related sheaves of rays which are equiangular.

Exercises

1. Construct a series of tangents to a conic, having given one focus and three tangents.

2. Given one focus, two tangents, and the point of contact on one of the tangents; construct a series of tangents to the conic.

3. Show that the circle circumscribed about any triangle, whose sides touch a parabola, passes through the focus of the parabola (Lambert, 1761).

4. Show that the four circles circumscribed about the four triangles formed by the sides of a complete quadrilateral pass through one and the same point.

5. If F and F_1 are the foci of a hyperbola, and any tangent meets the asymptotes in the points A and B, show that the quadrangle $AFBF_1$ can be inscribed in a circle.

II. *Any tangent to an ellipse, or to a hyperbola, bisects one of the angles between the focal radii drawn to its point of contact.*

Let T be the point of contact of the tangent, and F and F_1 the foci (Fig. 95). Let the given tangent meet the tangent at the vertex A in the point P. The lines PF and PF_1 are bisectors of the angles TFA and TF_1A, respectively (Art. 121). Consequently, P is equidistant from the sides of the triangle TFF_1, and hence on the bisector of one of the angles at T. Therefore, the tangent TP bisects one of the angles between the focal radii TF and TF_1, as was to be proved.

The circle with P as center and PA as radius is inscribed in the triangle TFF_1, in case the conic is a hyperbola; and is escribed to the triangle TFF_1, in case the conic is an ellipse.

III. *Any tangent to a parabola bisects one of the angles between the focal radius to the point of contact and the diameter through the point of contact.*

Thus, Fig. 96, the given tangent meets the tangent at the vertex A in the point P; and consequently PF bisects the angle AFT (Art. 121) and is perpendicular to PT (Art. 119, 3). The triangle

KFT is therefore isosceles, the angle at K is equal to the angle at T. The diameter through T is parallel to the axis; and conse-

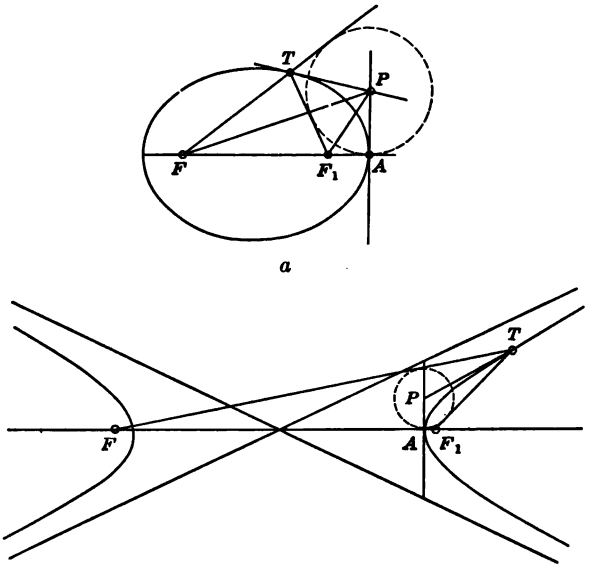


FIG. 95.

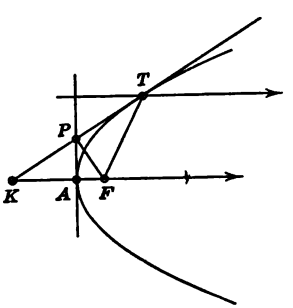


FIG. 96.

quently the tangent KT bisects one of the angles between the focal radius FT and the diameter through T.

NOTE.—The properties proved in II and III are usually called **reflection properties.** Consider the surface generated by the revolution of a conic around that axis upon which lie the real foci. A disturbance originating at one focus and travelling in straight lines will be reflected from the surface and proceed in straight lines toward, or away from, the other focus. Thus, rays of light emanating from the focus of a paraboloid of revolution will be reflected from the surface in parallel rays. In the case of an ellipsoid, the rays will converge toward the other focus; in the hyperboloid, they will be reflected away from the other focus.

IV. *The ratio of the distances of any point on a conic from a focus and from the corresponding directrix is constant.*

In Fig. 93, let F be a focus and f the corresponding directrix. If AB is any secant, meeting the conic in A and B and the directrix in P, then FP bisects the exterior angle at F of the triangle AFB. Consequently,

$$\frac{BP}{AP} = \frac{FB}{FA}.$$

If the feet of the perpendiculars from A and B upon the directrix are A_2 and B_2, respectively, then

$$\frac{BP}{AP} = \frac{BB_2}{AA_2}.$$

Hence,

$$\frac{FB}{BB_2} = \frac{FA}{AA_2},$$

which proves the proposition.

NOTE.—This property is often used in analytic geometry to define the conic sections. The constant ratio is called the **eccentricity** of the conic. The property itself was known to Pappus but was probably first used as a definition by Boscovich (1711–1787). For this reason, the eccentricity is sometimes called Boscovich's ratio.

Exercises

1. Show that the eccentricity of a parabola is unity (Art. 120).
2. If e is the eccentricity, show that:

$$e = \frac{\sqrt{a^2 - b^2}}{a}, \text{ for the ellipse;}$$

and

$$e = \frac{\sqrt{a^2 + b^2}}{a}, \text{ for the hyperbola.}$$

What is the significance of the fact that the common limit of these expressions as $a \doteq \infty$ is $e = 1$?

Show, also, that e is always less than unity, for the ellipse, and always greater than unity, for the hyperbola.

3. Show that the eccentricity of a circle is zero. How will a conic change in form as its eccentricity increases from 0 to ∞?

4. Given a focus, its corresponding directrix, and the eccentricity; construct a series of points on the conic.

5. Given a focus, its corresponding directrix, and one point; construct a number of points on the conic.

6. Show that, for the ellipse or the hyperbola, the distances from the center to a focus and from the center to the corresponding directrix are ae and a/e, respectively.

7. Given the foci and a point P on an ellipse, or a hyperbola, construct the tangent at P.

8. Given the focus, the axis, and a point P on a parabola; construct the tangent at P.

9. Given a right circular cylinder, show that the eccentricity of an ellipse, cut from the cylinder by a plane passing through a diameter of a circular section and making an angle x with this section, is $e = \sin x$.

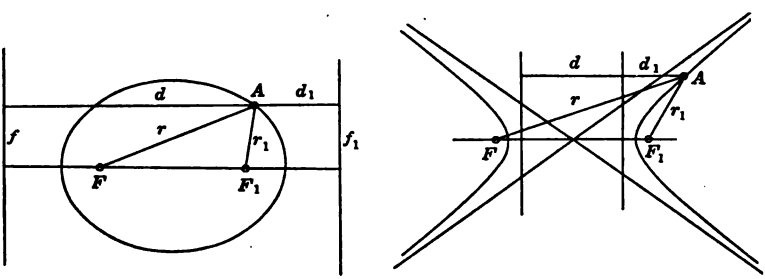


FIG. 97.

V. *The sum of the focal radii to any point on an ellipse is constant. The difference of the focal radii to any point on a hyperbola is constant.*

For, in Fig. 97, let r and r_1 be the focal radii to any point A, and d and d_1 the distances to the corresponding directrices. Since

an ellipse, or a hyperbola, is symmetric with respect to either axis, we have

$$\frac{r}{d} = \frac{r_1}{d_1} = e.$$

Therefore, also,

$$r \pm r_1 = e(d \pm d_1).$$

But for the ellipse $d + d_1$ (for the hyperbola $d - d_1$) is constant for all positions of the curve-point A, this number expressing the distance between the directrices. If A is taken at a vertex, we see that the sum of the lengths of the focal radii to any point on an ellipse is equal to the length of the major axis; and that the difference of the lengths of the focal radii to any point on a hyperbola is equal to the length of the major axis.

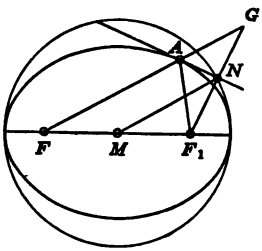


FIG. 98.

VI. *The feet of the perpendiculars from the foci of an ellipse, or of a hyperbola, upon a variable tangent lie upon a circle which is concentric with the curve and whose diameter is equal to the major axis of the curve.*

Let F and F_1 be the foci and A the point of contact of a variable tangent (Fig. 98). If the perpendicular from F_1 on the tangent at A meets the focal radius FA in G, then AG is equal to AF_1, since the tangent bisects the angle GAF_1. Hence FG = the length of the major axis = $2a$. If M is the center of the curve and N is the foot of the perpendicular upon the tangent, then $MN = \frac{1}{2}FG = a$. Consequently, the locus of N is a circle whose radius is a. This circle is called the **major auxiliary circle.**

A figure should be drawn and the proposition proved for the hyperbola.

In the case of the parabola, we have (Art. 119, 3):

The locus of the foot of a perpendicular from the focus of a parabola upon a variable tangent is the tangent at the vertex of the curve.

Exercises

1. Given the foci and the length of the major axis of an ellipse or of a hyperbola, construct the tangents from an exterior point to the curve and their points of contact.

Suggestions.—Let F and F_1 be the foci and P the point from which the tangents are to be drawn. With P as center draw a circle through the nearer focus, and with the other focus as center and the major axis as radius draw an arc cutting the first circle in the points D and D_1. The perpendicular bisectors of the segments FD and FD_1 are the required tangents, and the points in which these bisectors meet the lines joining D and D_1 to the farther focus are the required points of contact.

2. Given the focus and the directrix of a parabola, construct the tangents from an exterior point to the curve and their points of contact.

Suggestions.—Let F be the focus, f the directrix, and P the point from which the tangents are to be drawn. Draw the circle with PF as radius and P as center, meeting the directrix in D and D_1. The required tangents are the perpendicular bisectors of the segments FD and FD_1, and the points in which these bisectors meet the diameters through D and D_1 are the respective points of contact.

3. For an ellipse or a hyperbola, the four focal radii drawn from the extremities of any chord touch a circle whose center is the pole of the chord. State and prove the corresponding property for the parabola (Chasles, 1830).

4. Prove that the subnormal of a parabola is constant in length.

5. Show that the locus of the center of a circle touching two fixed circles is a conic. Discuss the problem for various positions of the fixed circles.

6. Construct a series of points of a conic having given one focus, the corresponding directrix, and one tangent.

7. If an ellipse and a hyperbola are confocal, show that they intersect at right angles.

VII. *The product of the perpendiculars from the foci of an ellipse, or of a hyperbola, upon a variable tangent is constant.*

The feet of the perpendiculars lie upon the major auxiliary circle by the preceding property. If the perpendiculars meet the auxiliary circle again in the points N and M_1 (Fig. 99), then NM and N_1M_1 are diameters of this circle. Hence we have, $F_1N_1 \cdot FM = F_1N_1 \cdot F_1N = AF_1 \cdot F_1B = (AO - OF)(AO + OF) = a^2 - \overline{OF}^2 = b^2$ (Art. 119, exercise 3).

The student should prove the proposition for the hyperbola, making use of Art. 119, exercise 4.

VIII. *The locus of the vertex of a right angle whose sides touch an ellipse or a hyperbola is a circle concentric with the conic.*

Thus, if R is the vertex of a right angle whose sides touch the ellipse (Fig. 99), and S and T are the feet of perpendiculars from the foci upon one side, then

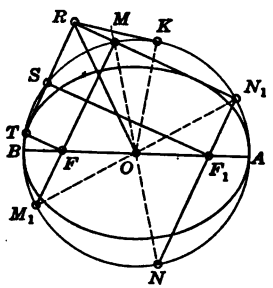


FIG. 99.

$$F_1S \cdot FT = N_1R \cdot MR = b^2 = \overline{RK}^2,$$

where RK is the length of the tangent from R to the major auxiliary circle. Hence

$$\overline{OR}^2 = \overline{OK}^2 + \overline{RK}^2 = a^2 + b^2.$$

The locus of R is, therefore, the circle whose center is O and whose radius is the constant distance OR. This circle is called the **director circle** for the conic in question.

The student should construct a figure and prove the proposition for the hyperbola. The radius of the director circle for the hyperbola is $\sqrt{a^2 - b^2}$, and consequently the hyperbola has a real director circle only when $b < a$; that is, only when the angle between the asymptotes, within which the curve lies, is acute.

Exercises

1. Prove that the locus of the vertex of a right angle whose sides touch a parabola is the directrix.

2. Show that the directrices of all parabolas which can be inscribed in a given triangle pass through the orthocenter of the triangle and the foci lie on the circumscribed circle.

3. If perpendiculars are dropped from any point of a circle upon the sides of an inscribed triangle, prove that the feet of these perpendiculars lie upon a straight line which is the tangent at the vertex of one of the parabolas inscribed in the triangle (Simson line, or Wallace line).

4. If P is a point outside an ellipse or a hyperbola, show that the lines joining P to the foci are equally inclined to the tangents drawn from P.

5. A conic is inscribed in a given triangle ABC and F is one focus. Construct the other focus.

CHAPTER XIV

PROJECTIVELY RELATED PRIMITIVE FORMS OF THE SECOND KIND

123. Primitive Forms of the Second Kind. Incident Elements.—The field of points, the field of rays, the bundle of rays, and the bundle of planes constitute a group of primitive forms such that each form can be derived from one of the others by projection or section. Thus, the projector of a field of rays is a bundle of planes. Again, the dual, or reciprocal, of one of these forms is a form belonging to the same group. The forms above enumerated are called **primitive forms of the second kind.** Primitive forms of the second kind are two-dimensional forms (Art. 15).

A field of points and a field of rays can be superposed, thus forming a **plane**; and a bundle of rays and a bundle of planes can be superposed, forming a **bundle.** The plane and the bundle are reciprocal, or dual, figures in space; and the one is a section of the other. A plane contains all the points and lines lying on it; and a bundle is composed of all the lines and planes passing through its vertex.

If, in a plane, the point A lies on the line a, the two elements A, a are said to be **incident** elements. Similarly, in a bundle, if the line a lies in the plane α, the two elements a, α are **incident** elements. Thus, in a plane, a point is incident to all the lines passing through it, and a line is incident to all the points lying on it. In a bundle, a line is incident to all the planes through it, and a plane is incident to all the lines of the bundle lying in it (cf. Art. 14).

124. Perspective Position of Planes and Bundles.—A plane and a bundle are in perspective position if the plane is a section of the bundle, each element of the plane then lies on its corresponding element of the bundle.

Two planes are in perspective position if each is a section of the same bundle. Corresponding elements in the planes then lie on the same element of the bundle.

Two bundles are in perspective position if each is a projector of

the same plane. Corresponding elements in the bundles then have in common the same element of the plane.

A series of planes and bundles, each of which is in perspective position with the one following it, constitutes a **chain of perspectivity.** The first and the last of a chain of perspectivity are correlated, element to element, but are not, in general, in perspective position.

If two planes, or two bundles, are in perspective position, or are correlated by means of a chain of perspectivity connecting them, corresponding elements are always of the same kind; that is, point corresponds to point and line to line, or line to line and plane to plane.

125. Orthogonally Correlated Bundles.—Two bundles can be correlated so that to any element of either (line or plane) corresponds that element of the other (plane or line) which is perpendicular to it. If S and S_1 are the centers of the bundles, then two lines a and b of S correspond to the planes α_1 and β_1 of S_1 which are respectively perpendicular to a and b. The plane (ab) of S corresponds to the line $(\alpha_1\beta_1)$ of S_1; and it is clear that (ab) is perpendicular to $(\alpha_1\beta_1)$. Two bundles correlated in this manner are said to be **orthogonally correlated.**

If we cut each of two orthogonally correlated bundles by planes, the sections so obtained are correlated, element to element, but are neither in perspective position nor are they the first and last of a chain of perspectivity. In fact they are so correlated that point corresponds to line and line to point.

126. Definition of Projective Relationship.—To avoid circumlocution, the plane and the bundle will be called **forms of the second kind.**

Two correlated forms of the second kind are projectively related if incident elements in either always correspond to incident elements in the other.

Thus two correlated forms of the second kind are projectively related if they are in perspective position, or if they are the first and last of a chain of perspectivity. Again, two bundles are projectively related if they are orthogonally correlated; two planes are projectively related if they are sections of orthogonally correlated bundles.

127. Definition of Collineation.—*If two forms of the second kind are projectively related and corresponding elements are of like kinds, the forms are said to be collinearly related and the projective relationship is called a collineation.*

Thus, if two forms are in perspective position, or are connected by a chain of perspectivity, they are collinearly related.

128. Definition of Duality.—*If two forms of the second kind are projectively related and corresponding elements are of unlike kinds, the forms are said to be reciprocally related and the projective relationship is called a duality.*

Thus if two bundles are orthogonally correlated, they are reciprocally related. The plane sections of two orthogonally correlated bundles are reciprocally related planes.

129. Consequences of the Foregoing Definitions.—

1. *Two forms of the second kind, each collinearly related to a third form, are collinearly related to each other.*

2. *Two forms of the second kind, each reciprocally related to a third form, are collinearly related to each other.*

3. *Two forms of the second kind, one reciprocally related and the other collinearly related to a third form, are reciprocally related to each other.*

4. *If two forms of the second kind are projectively related, any primitive form of the first kind in either corresponds to a primitive form of the first kind in the other.*

Thus, any point-row in one of two collinearly related planes corresponds to a point-row in the other. If the planes are reciprocally related, any point-row in one corresponds to a sheaf of rays in the other. In reciprocally related bundles, a sheaf of rays in one corresponds to a sheaf of planes in the other.

Primitive forms of the first kind that correspond to each other in projectively related forms of the second kind are called **corresponding primitive forms of the first kind.**

130. Fundamental Theorem.—Theorem XV.—*If two forms of the second kind are projectively related, then corresponding primitive forms of the first kind are projectively related to each other.*

Suppose that α and α_1 are collinearly related planes, and that u and u_1 are corresponding point-rows (Fig. 100). We are to show that, in the collineation existing between the planes, any

four harmonic points of u correspond to four harmonic points of u_1. Let $ABCD$ be any harmonic range on u defined by a complete quadrangle $KLMN$ in α, and let corresponding points in α_1 be denoted by subscripts, so that A_1 corresponds to A, B_1 to B, and so on. Since the planes are projectively related, collinear points in one correspond to collinear points in the other,

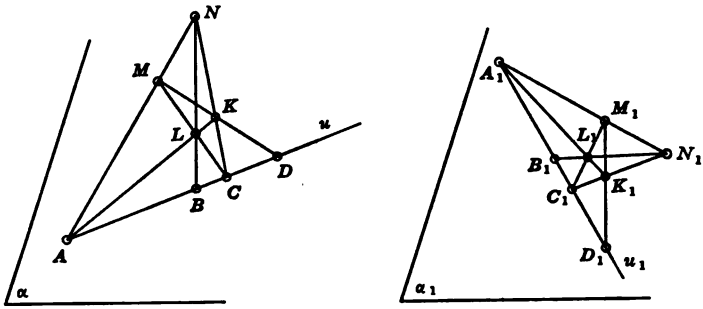


FIG. 100.

and concurrent lines to concurrent lines. It follows at once that $A_1B_1C_1D_1$ is a harmonic range on u_1 defined by the complete quadrangle $K_1L_1M_1N_1$. Since $ABCD$ is any harmonic range on u, and u is any point-row in α, it follows that any two corresponding point-rows are projectively related.

If N and N_1 are the centers of any two corresponding sheaves of rays, then

$$N \overline{\overline{\wedge}}\, u \,\overline{\wedge}\, u_1 \,\overline{\overline{\wedge}}\, N_1,$$

where u and u_1 are corresponding sections of N and N_1 respectively. Hence, $N \,\overline{\wedge}\, N_1$.

If α and α_1 are reciprocally related planes, a harmonic range and its defining quadrangle in α corresponds to a pencil of rays and a quadrilateral in α_1. Since, in the duality existing between the planes, collinear points correspond to concurrent lines, and concurrent lines to collinear points, it is an easy matter to show that the corresponding pencil is harmonic, and hence that corresponding primitive forms of the first kind are projectively related. The student should complete the proof.

If the projectively related forms are bundles, each can be cut by a plane, and the proof of the theorem thus reduced to the cases already considered.

Finally, if a plane is projectively related to a bundle, the theorem still holds for these forms, since the given plane is projectively related to any plane section of the given bundle.

131. Determination of Projective Relationship.—*A projective relationship between two forms of the second kind is uniquely determined by choosing any four elements of like kind in one, no three of which belong to the same primitive form of the first kind, to correspond*

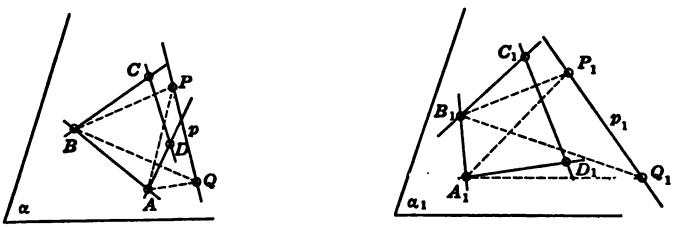


FIG. 101.

to four arbitrarily chosen elements of like kind in the other, no three of which belong to any primitive form of the first kind.

For example, a collineation is established between two planes when the vertices of any quadrangle $ABCD$ in one are chosen to correspond respectively to the vertices $A_1B_1C_1D_1$ of any quadrangle in the other (Fig. 101). For then

$$A(BCD\text{——}) \;\overline{\wedge}\; A_1(B_1D_1C_1\text{——})$$

and

$$B(ACD\text{——}) \;\overline{\wedge}\; B_1(A_1C_1D_1\text{——}),$$

the three pairs of corresponding rays, in each case, being just sufficient to determine the projectivity between the corresponding sheaves of rays. If P is any point in α, we can construct the rays A_1P_1 and B_1P_1 in α_1 which correspond respectively to the rays AP and BP in α. Hence, P_1 is uniquely determined.

If p is any line in α and P and Q are any two points on p, we can construct the points P_1 and Q_1 in α_1; and hence the line p_1 is uniquely determined. The collineation is therefore completely determined.

A collineation is also completely determined between two planes by choosing the sides of any complete quadrilateral in one to correspond respectively to the sides of any complete quadrilateral in the other.

A duality is completely determined between two planes by choosing the vertices of any complete quadrangle in one to correspond respectively to the sides of a complete quadrilateral in the other. The student should construct a figure and supply the details of the proof.

Since projectively related bundles can be cut in projectively related planes, it follows that: A collineation is determined between two bundles if the edges (faces) of any complete 4-edge (4-face) in one are chosen to correspond respectively to the edges (faces) of any complete 4-edge (4-face) in the other; and a duality is determined between two bundles if the edges (faces) of a complete 4-edge (4-face) in one are chosen to correspond respectively to the faces (edges) of any complete 4-face (4-edge) in the other.

Exercises

1. If two planes are collinearly related, prove that a conic in either corresponds to a conic in the other. If the planes are reciprocally related, what corresponds to a conic in one?

2. If a plane is projectively related to a bundle, what corresponds to a conic in the plane? There are two cases.

3. If two bundles are orthogonally correlated, show that corresponding elements intersect upon a sphere whose center bisects the line joining the centers of the given bundles.

4. If two bundles are reciprocally related, show that the surface upon which lie the points of intersection of corresponding elements is, in general, cut by any plane of either bundle in a conic. Examine the section of this surface by that plane of either bundle which corresponds to the line joining the centers of the bundles.

5. A plane is divided into squares by lines drawn parallel to two perpendicular axes as in analytic geometry. If the origin, the ideal point on each axis, and the point whose coördinates are (1, 1) are made to correspond to the vertices of any complete quadrangle in a second plane, show how to divide the second plane into quadrangles corresponding to the squares in the first plane. If the coördinates of a point in the first plane are any two rational numbers, construct the

corresponding point in the second plane. Can the construction be effected if the coördinates are irrational numbers?

6. Two bundles are orthogonally correlated and concentric. Show that the duality determined upon a plane section of the bundles is involutoric. If the bundles are not concentric, will the duality on a plane section be involutoric? Show that, in the second case, the locus of points which lie on their corresponding lines is, in general, a circle.

132. Projectivities in a Form of Second Kind.—If two forms of the second kind are projectively related, we shall say that a projectivity exists between them, and we shall regard this projectivity as a transformation which transforms one form into the other. If the forms can be superposed, that is, if they are both planes or both bundles, the projectivity transforms their common base into itself. Thus, two collinearly related and superposed planes constitute a collineation which transforms their common plane into itself, replacing each point by another (or in particular cases by the same) point, and, in general, each line by another line. A duality in a plane results from superposing two reciprocally related planes, and replaces each point of the common plane by a line of that plane, and each line by a point.

In the succeeding articles we shall study, in some detail, various types of collineations.

133. The Perspectivity.—If two collinearly related planes are in perspective position, they must have their line of intersection as a self-corresponding line. For every point of this line, considered as a point in either plane, coincides with its corresponding point in the other plane.

Likewise, the line joining the centers of two bundles in perspective position is a self-corresponding line. For every plane through this line, considered as a plane of either bundle, coincides with its corresponding plane of the other bundle.

Conversely:

If two planes are collinearly related and have a self-corresponding line, they are in perspective position, or else they are superposed and have in common a sheaf of rays.

If two bundles are collinearly related and have a self-corresponding ray, they are in perspective position, or else they are superposed and have in common a sheaf of rays.

On the left, if ABC and $A_1B_1C_1$ are any two corresponding triangles and u is the self-corresponding line, then corresponding sides of these triangles must meet on u. The triangles are then in perspective position (Art. 17), and lines joining corresponding vertices intersect in a point U. Any fourth point D is joined to its corresponding point D_1 by a line through U, since the quadrangles $ABCD$ and $A_1B_1C_1D_1$ are in perspective position (Art. 20). The planes are then either sections of the same bundle U, and consequently in perspective position, or else they are superposed and have in common the sheaf of rays U. In the latter case we shall say they constitute a **perspectivity** in their common plane.

The theorem on the right is the space-dual of the theorem on the left. The bundles are then either the projectors of the same

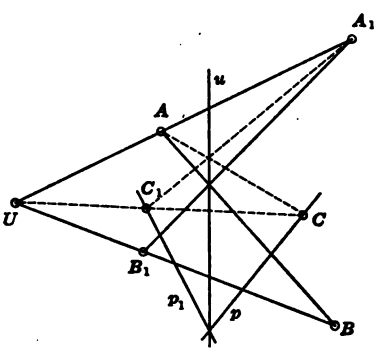


FIG. 102.

plane, and consequently in perspective position, or else they are superposed and have in common a sheaf of rays. In the latter case we shall say they constitute a **perspectivity** in their common bundle.

A perspectivity in a plane has a self-corresponding point-row, called the **axis of perspectivity,** and a self-corresponding sheaf of rays whose center is called the **center of perspectivity** (*cf.* Art. 19).

A perspectivity in a bundle has a self-corresponding sheaf of planes and a self-corresponding sheaf of rays.

134. Construction of a Perspectivity in a Plane.—*Given the axis and the center of a perspectivity and one pair of corresponding points,*

or one pair of corresponding rays, to construct as many pairs of corresponding elements as may be desired.

Let u and U be respectively the given axis and the given center, and A, A_1 the given pair of corresponding points (Fig. 102). If B is any point in the plane, the lines AB and A_1B_1 must meet on u, and the line BB_1 must pass through U. Hence the point B_1, corresponding to B, is immediately constructed.

If p is any ray in the plane and C is any point on p, we can construct C_1, corresponding to C; and then p_1 must pass through C_1 and meet p on the axis u.

Exercises

1. Work out the details of the proof for the theorem on the right in Art. 133.

2. Construct a number of corresponding elements having given the axis and the center of a perspectivity in the plane and one pair of corresponding rays.

3. Given one pair of corresponding points, construct pairs of corresponding elements in a perspectivity: (*a*) when the axis is the ideal line in the plane and the center is an actual point; (*b*) when the axis is an actual line and the center is ideal; (*c*) when both axis and center are ideal.

135. The Invariant of a Perspectivity in a Plane.—Let u and U be the axis and the center of a perspectivity, and A, A_1 any pair of corresponding points (Fig. 103). If the line UAA_1 meets the axis in O, we can prove that the cross-ratio (Art. 28) of the range $UAOA_1$ is constant for all positions of A. If B and B_1 are a pair of corresponding points, then the ranges $UAOA_1$ and $UBMB_1$ are sections of the same pencil whose center is a point Q on the axis. It is easy to show that:

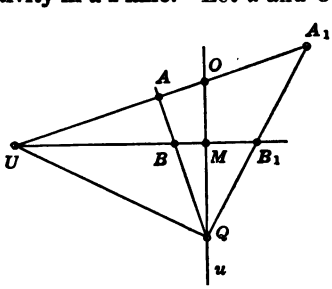


FIG. 103.

$$(1) \qquad \frac{UA}{AO} \div \frac{UA_1}{A_1O} = \frac{\sin UQA}{\sin AQO} \div \frac{\sin UQA_1}{\sin A_1QO} \text{ (cf. Art. 32, exercise 4).}$$

But the cross-ratio of the range $UBMB_1$ is equal to the same constant, since this constant depends only upon the angles at Q: The right hand member of (1) is the cross-ratio of the pencil $Q(UAOA_1)$. This cross-ratio is clearly the same for any other pencil of which $UAOA_1$ is a section. We conclude, therefore, that the cross-ratio is the same for all ranges formed like $UAOA_1$; and that this constant is equal to the cross-ratio of all pencils formed like $Q(UAOA_1)$. This constant is called the **invariant** of the perspectivity.

136. The Harmonic Perspectivity, or Involution, in a Plane.—The perspectivity whose invariant is -1 is called a **harmonic perspectivity,** or an **involution.** In this case, the ranges $UAOA_1$, $UBMB_1$, etc., are harmonic (Art. 28). The lines AB_1 and A_1B (Fig. 103) meet on the axis in a point T, and consequently A and A_1 correspond doubly. In the same way, the points of any pair correspond to each other doubly. There is an involution along each line through the center whose foci are the center and the point in which the line meets the axis u; and there is an involution about each point of the axis whose focal rays are the axis and the ray joining the point to the center U. A harmonic perspectivity is thus involutoric.

The collineation in a plane determined by choosing the vertices A, A_1, B, B_1 of a quadrangle to correspond respectively to the points A_1, A, B_1, B is an involution.

137. Limiting Lines in a Collineation.—If two planes are collinearly related, the ideal line in either corresponds, in general, to an actual line in the other. The actual line in either plane which corresponds to the ideal line in the other is called the **limiting line** in its plane.

With the aid of the limiting lines we can place two collinearly related planes in perspective position. Thus (Fig. 104), let α and α_1 be collinearly related planes, and suppose c and d_1 are respectively the limiting lines in α and α_1. To the ideal point J on c corresponds the ideal point J_1 on d_1, and hence to the sheaf of rays whose center is J corresponds the sheaf of rays whose center is J_1. Let a, b be a pair of parallel rays in α not belonging to the sheaf J. The corresponding pair a_1, b_1 in α_1 must intersect on d_1. Suppose the rays a, b determine a segment r units in length upon c. In the

plane α_1 select the two rays e_1 and f_1 of the sheaf J_1 upon which the rays a_1, b_1 determine segments r units in length. We now have a quadrilateral $a_1b_1e_1f_1$ in α_1 whose vertices $A_1B_1C_1D_1$ correspond to the vertices of a parallelogram $ABCD$ in α each to each. The two planes can be placed so that the points A, B, J coincide with their corresponding points A_1, B_1, J_1. The projectively related

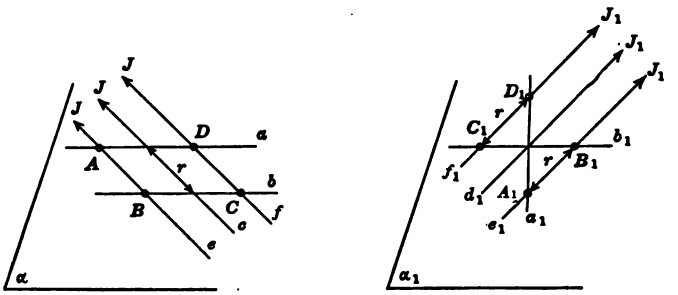


FIG. 104.

point-rows e and e_1 will then coincide throughout and the planes are consequently in perspective position.

The point-rows f and f_1 can also be placed in coincidence; hence two collinearly related planes whose ideal lines do not correspond to each other can always be placed in perspective position in at least two ways.

Exercises

1. Given the center and the axis of a perspectivity in a plane and one pair of corresponding points, P and P_1. If P is made to describe any conic, construct the conic described by P_1.

2. Construct the limiting lines of a perspectivity in a plane. Show that the limiting lines are necessarily parallel to the axis of perspectivity.

3. If the points $ABCD$ of a plane correspond to the points $BADC$, each to each, show that the collineation so determined is an involution.

4. Two collinearly related planes α and α_1 are in perspective position. If α is rotated around their common line μ, without disturbing the relative position of its elements, show that the center of the bundle of which α and α_1 are sections will describe a circle whose center is in

α_1 and corresponds to an ideal point in α, the plane of the circle being perpendicular to μ.

5. Given two collinearly related planes whose ideal lines do not correspond to each other, find a sheaf of rays in one to which corresponds a congruent sheaf of rays in the other.

Suggestions.—Choose two right angles in one plane whose sides a, b and c, d do not form a parallelogram. Let A, B, C, D be the ideal points on a, b, c, d respectively, and A_1, B_1, C_1, D_1 be the corresponding points on the limiting line in the other plane. Draw circles on the segments A_1B_1 and C_1D_1 as diameters. These circles intersect in two points P_1 and Q_1 (why?). The sheaves of rays whose centers are P_1 and Q_1 correspond to sheaves in the other plane to which they are also congruent, each to each.

138. The Affinity.—Two collinearly related planes whose ideal lines correspond to each other are said to be **affinately related** to each other, and the correspondence existing between the two planes is called an **affinity.**

An affinity can be established between two planes by choosing the sides of any triangle in one to correspond to the sides of any triangle in the other, each to each. For these triangles, together with the ideal line in each plane, form two complete quadrilaterals whose sides correspond, each to each. A collineation is thus established between the two planes (Art. 131) such that the ideal line in one corresponds to the ideal line in the other.

139. Fundamental Property of an Affinity.—Corresponding point-rows in affinately related planes are similarly projective (Art. 61), that is, corresponding segments bear a constant ratio to each other. The following fundamental property is a consequence of this fact.

In affinately related planes, the ratio between the areas of corresponding figures is constant.

For a parallelogram in one plane must correspond to a parallelogram in the other, since the ideal line in one corresponds to the ideal line in the other. Suppose (Fig. 105) the parallelograms $ABCD$ and $EFGH$ in α correspond respectively to the parallelograms, $A_1B_1C_1D_1$ and $E_1F_1G_1H_1$ in α_1. It follows that the parallelogram $KLMN$, constructed as in the figure, corresponds to the parallelogram $K_1L_1M_1N_1$. Since the areas of parallelograms hav-

ing the same altitude are to each other as the lengths of corresponding sides, we have the following proportions:

$$\frac{ABCD}{KLMN} = \frac{AB}{LK}, \qquad \frac{A_1B_1C_1D_1}{K_1L_1M_1N_1} = \frac{A_1B_1}{L_1K_1},$$

$$\frac{EFGH}{KLMN} = \frac{EF}{LM}, \qquad \frac{E_1F_1G_1H_1}{K_1L_1M_1N_1} = \frac{E_1F_1}{L_1M_1},$$

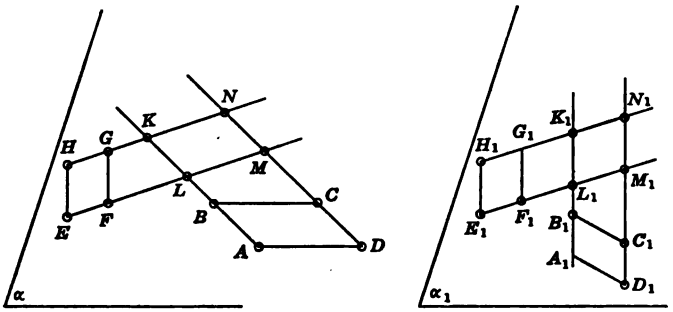


FIG. 105.

and hence,

$$\frac{ABCD}{EFGH} = \frac{AB}{EF}, \qquad \frac{A_1B_1C_1D_1}{E_1F_1G_1H_1} = \frac{A_1B_1}{E_1F_1}.$$

But,

$$\frac{AB}{EF} = \frac{A_1B_1}{E_1F_1},$$

and therefore,

$$\frac{ABCD}{A_1B_1C_1D_1} = \frac{EFGH}{E_1F_1G_1H_1} = m \text{ (constant)}.$$

Since a triangle can be regarded as half a parallelogram, we see that the areas of corresponding triangles are to each other in the same constant ratio m.

Corresponding polygons can be divided into series of corresponding triangles and hence, by composition of ratios, their areas are to each other in the same constant ratio m.

Finally, the areas of corresponding curves can be regarded as the limits of corresponding series of circumscribed, or inscribed,

polygons and hence bear the same constant ratio to each other. We conclude, therefore, that if A and A_1 represent the areas of any two corresponding figures in two affinately related planes, we always have

$$\frac{A}{A_1} = m.$$

140. Corresponding Conics in Affinately Related Planes.—Corresponding conics in affinately related planes are always of the same kind, since the ideal points on one conic must correspond to the ideal points on the other. Thus, ellipses can only correspond to ellipses (or to circles), parabolas to parabolas, and hyperbolas to hyperbolas.

Conversely, two conics of the same kind can be correlated point to point in many ways so that they shall be corresponding conics

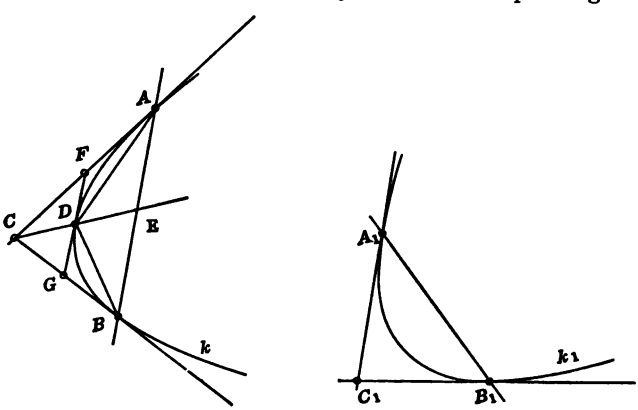


FIG. 106.

in affinately related planes. Thus, suppose k and k_1 (Fig. 106) are two parabolas, correlated point to point, and that A and B correspond respectively to A_1 and B_1. The tangents at these pairs of points intersect in the points C and C_1. The triangles ABC and $A_1B_1C_1$ serve to determine an affinity between the planes of k and k_1 (Art. 138) in which to the parabola k must correspond a parabola k' touching A_1C_1 in A_1 and B_1C_1 in B_1. But k' coincides with k_1, since it has three tangents and the points of contact on two of them in common with k_1.

141. The Area of a Parabolic Segment.—*The area of any parabolic segment is two-thirds the area of the triangle whose sides are the chord of the segment and the tangents at the extremities of the chord.*

For, if (AB) represents the area of the parabolic segment whose chord is AB (Fig. 106), we have the following proportion:

$$\frac{(AB)}{(A_1B_1)} = \frac{A\ B\ C}{A_1B_1C_1}.$$

But we can establish an affinity between the planes of k and k_1 by choosing any other triangle formed like ABC to correspond to $A_1B_1C_1$. Let D be the point where the diameter through C meets the parabola, and let FG be the tangent at D. Now in the affinity determined by the triangles ADF and $A_1B_1C_1$ we have

$$\frac{(AD)}{(A_1B_1)} = \frac{A\ D\ F}{A_1B_1C_1}.$$

Similarly,

$$\frac{(DB)}{(A_1B_1)} = \frac{D\ B\ G}{A_1B_1C_1}.$$

Hence,

$$\frac{(AB)}{ABC} = \frac{(AD)}{ADF} = \frac{(DB)}{DBG} = m.$$

The constant m can be found by means of the equation

$$(AB) = (AD) + (DB) + ADB. \qquad (1)$$

For, since D bisects CE (Art. 75, 5), $ADB = \frac{1}{2}ABC$. Also,

$$FCG = ADF + DBG = \tfrac{1}{4}ABC.$$

Hence, from (1),

$$m \cdot ABC = m \cdot ADF + m \cdot DBG + \tfrac{1}{2}ABC,$$

or

$$m \cdot ABC = \frac{m}{4} \cdot ABC + \tfrac{1}{2}ABC.$$

Therefore, $m = \frac{2}{3}$ and $(AB) = \frac{2}{3}ABC$.

142. The Theorem of Apollonius.—When two ellipses, or two hyperbolas, are corresponding conics in two affinately related planes, any diameter of the one must correspond to a diameter of the other. For a system of parallel chords of the one corresponds to a system of parallel chords of the other. The mid-points of the one system must correspond to the mid-points of the other system, since the mid-point of any chord is the harmonic conjugate

of the ideal point on that chord with respect to the curve points. Hence diameters correspond to diameters. It follows also that a pair of conjugate diameters of the one curve always corresponds to a pair of conjugate diameters of the other.

Suppose k and k_1 are two ellipses in the same or in different planes (Fig. 107), and suppose AC, BD and A_1C_1, B_1D_1 are pairs of conjugate diameters of k and k_1 respectively. In the affinity determined by the triangles ABC and $A_1B_1C_1$, the two ellipses must correspond point to point, since the conic corresponding to k must be an ellipse having A_1C_1 and B_1D_1 for a pair of conjugate diameters and it must pass through A_1, B_1, and C_1 and hence coincides with k_1.

Theorem of Apollonius.—*The area of any parallelogram inscribed in an ellipse and whose diagonals are conjugate diameters is* $2ab$, *where* a *and* b *are the lengths of the semi-axes.*

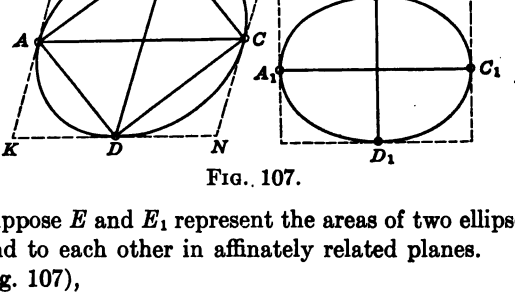


FIG. 107.

For, suppose E and E_1 represent the areas of two ellipses which correspond to each other in affinately related planes.
Then (Fig. 107),

$$\frac{A\,B\,C\,D}{A_1B_1C_1D_1} = \frac{E}{E_1}, \text{ or } ABCD = \frac{E}{E_1} \cdot (A_1B_1C_1D_1).$$

The right-hand member of this equation is constant for all parallelograms formed like $ABCD$. Therefore, these parallelograms all have the same area. But the parallelogram whose diagonals are the axes of the ellipse is formed like $ABCD$ and its area is $2ab$, where a and b are the lengths of the semi-axes. Hence the theorem.

The theorem of Apollonius is at once extended to parallelograms circumscribed about an ellipse and whose sides are parallel

to pairs of conjugate diameters, since the area of such a parallelogram is twice the area of the corresponding inscribed parallelogram. In the figure, the area of $KLMN$ is twice the area of $ABCD$. Hence:

The area of any parallelogram circumscribed about an ellipse and whose sides are parallel to a pair of conjugate diameters is $4ab$, where a and b are the lengths of the semi-axes.

From either of the above theorems, we can at once determine the area of an ellipse. For we can establish an affinity between the plane of the ellipse and the plane of any circle so that the two curves shall correspond point to point. If E and P represent respectively the area of the ellipse and the area of a parallelogram circumscribed about the ellipse whose sides are parallel to a pair of conjugate diameters, while E_1 and P_1 stand for the area of the circle and the area of the circumscribed square which corresponds to the parallelogram P, then

$$\frac{E}{E_1} = \frac{P}{P_1}.$$

But $P_1 = 4r^2$ (r being the radius of the circle), $P = 4ab$, and $E_1 = \pi r^2$. Substituting in the above equation, we have

$$E = \pi ab.$$

Exercises

1. Show that the area of any triangle inscribed in a parabola is twice the area of the triangle whose sides touch the parabola at the vertices of the inscribed triangle.

2. Two planes are superposed, determine the axis and center of a perspectivity between them which will transform a given triangle in one into an equilateral triangle in the other.

Suggestions.—Let ABC be the given triangle and M the mid-point of the side AC. Draw the median BM and parallels to it through A and C. Erect any perpendicular to BM and let it meet the parallels in A_1 and C_1. Construct an equilateral triangle on A_1C_1 as one side and let B_1 be the third vertex. The triangles ABC and $A_1B_1C_1$ are in perspective position.

3. Show that the perspectivity determined in the last exercise is an affinity.

4. Show that in two superposed and perspectively related planes any conic is transformed into itself provided it passes through one

pair of corresponding points and the center and axis of perspectivity are pole and polar line with respect to the conic. Show also that the perspectivity is necessarily an involution.

5. If two hyperbolas correspond to each other in affinately related planes, show that the asymptotes of one correspond to the asymptotes of the other.

6. If k and k_1 are two hyperbolas in the same plane, or in different planes, and we choose the triangle formed by the asymptotes and any third tangent of the one to correspond to the triangle formed by the asymptotes and any third tangent of the other each to each, show that an affinity is determined between the planes in which the hyperbolas correspond point to point.

143. The Similitude.—If in two affinately related planes, the corresponding ideal point-rows are congruent; that is, if they can be superposed so as to coincide point to point, the planes are said to be **similar**, and the collineation existing between them is called a **similitude.**

When two similar planes are superposed, the similitude existing between them is called **direct** or **inverse** according as their ideal point-rows are directly or oppositely projective (Art. 40).

When two similar planes are superposed and in perspective position; that is, when their ideal point-rows coincide point to point, the center of perspectivity is called the **center of similitude,** and the collineation is called a **perspective similitude.**

144. Properties of the Similitude.—

1. *In a similitude, corresponding angles are equal.*

For, if P corresponds to P_1, and the ideal points A and B correspond respectively to the ideal points A_1 and B_1, the angle APB is equal to the angle $A_1P_1B_1$, since the planes can be superposed so that the points A and B coincide respectively with A_1 and B_1.

2. *In a similitude, corresponding figures are similar figures.*

For, if ABC and $A_1B_1C_1$ are corresponding triangles, they are equiangular by property 1, and consequently similar.

Two corresponding polygons can be divided into sets of corresponding triangles and hence are similar polygons.

Corresponding curves can be considered as the limits of sets of corresponding inscribed, or circumscribed, polygons and are consequently similar curves. In particular, the circles in one plane correspond to circles in the other.

3. *If a collineation between two planes transforms the circles in one plane into circles in the other, the collineation is a similitude.*

For the ideal line in one plane must correspond to the ideal line in the other, otherwise the circles in one plane which cut the limiting line in that plane would be transformed into hyperbolas in the other plane, contrary to hypothesis. The collineation is, therefore, an affinity, and consequently to each pair of perpendicular (conjugate) diameters of any circle must correspond a pair of perpendicular diameters of the corresponding circle (Art. 142). Hence the collineation is a similitude.

4. *A similitude is determined between two planes by choosing any triangle in one to correspond to a similar triangle in the other.*

For, in the affinity determined by these two triangles, the corresponding ideal point-rows are congruent.

5. *Any two circles in a plane correspond to each other in two definite perspective similitudes. One of these is direct and corresponding parallel radii are drawn in the same direction; the other is inverse and corresponding parallel radii are drawn in opposite directions.*

Let O and O_1 be the centers of any two circles k and k_1, respectively (Fig. 108). Draw any pair of parallel diameters as AC and A_1C_1 and let AA_1 and OO_1 meet in S. The similar triangles AOS and A_1O_1S determine a direct perspective similitude which transforms k into k_1. If AC_1 meets OO_1 in S', then the triangles AOS' and C_1O_1S' determine an inverse perspective similitude which likewise transforms k into k_1.

S is called the **direct** center of similitude and S', the **inverse** center of similitude.

145. Inverse Points. Radical Axis.—Any secant through a center of similitude S (Fig. 108) cuts a pair of corresponding circles k and k_1 in two pairs of corresponding points A, A_1 and B, B_1. Two points cut by the same secant, one from each circle, but which do not correspond to each other are called **inverse points.** Thus B and A_1 are inverse points. So also are A and B_1.

If the points on k are correlated to their respective inverse points on k_1 a perspectivity is determined in the plane whose center coincides with S and which transforms k into k_1. The axis of this perspectivity is called the **radical axis** of the two circles.

Tangents at corresponding inverse points intersect upon the radical axis. Thus the radical axis of the two circles k and k_1 is the line TQ.

The segments of the tangents to two circles contained between any point of the radical axis and the points of contact are equal in length.

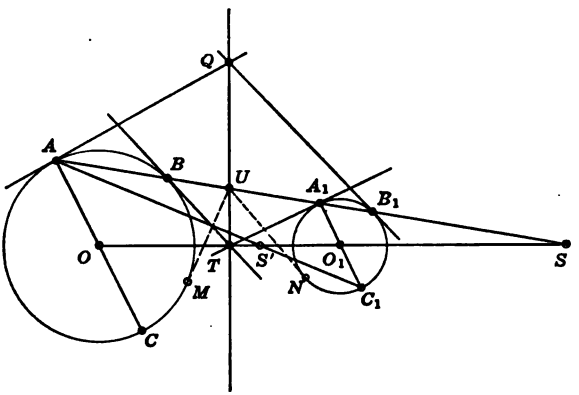


FIG. 108.

Let any secant through S meet the radical axis in the point U and the points of contact of tangents from U be M and N. The tangents at A and A_1 are parallel to each other. So also are the tangents at B and B_1. Hence, the triangles TUA_1 and QUA are similar. Likewise, the triangles TUB and QUB_1 are similar. Therefore,

$$\frac{UA_1}{TA_1} = \frac{UA}{AQ} \text{ and } \frac{UB_1}{QB_1} = \frac{UB}{BT}.$$

But TBA_1 and AQB_1 are isosceles triangles, since each has a pair of equal angles, and hence $TA_1 = BT$ and $AQ = QB_1$. It follows that,

$$UA_1 \cdot UB_1 = UA \cdot UB,$$

and therefore,

$$UN = UM.$$

146. The Congruence.—A special similitude is the **congruence** in which corresponding similar triangles are congruent triangles. Two congruent planes can be superposed so as to coincide throughout. A congruence is either direct or inverse.

Just as the affinity is a special collineation and the similitude is a special affinity, so is the congruence a special similitude.

Exercises

1. Show that a circle can be drawn to touch two given circles in a pair of inverse points.

2. If a circle is drawn to touch two given circles, show that the line joining the points of contact will pass through a center of similitude.

3. Three circles, taken two and two, determine three radical axes. Show that these radical axes meet in a point. This point is called the **radical center** of the three circles.

4. Two given circles are external to each other and cut the line joining their centers in the points A, B and C, D. Any circle whose center is a point O on the radical axis of the two given circles and whose radius is equal to the length of the tangent from O to either of the given circles will cut the given circles orthogonally and the line $ABCD$ in two points which separate harmonically both A, B and C, D.

5. The radical axis of any two circles, each of which touches two given circles, passes through a center of similitude of the given circles.

6. One circle is inside another but not concentric with it. Construct the centers of similitude.

7. Two circles intersect. Construct the centers of similitude and show that the radical axis is the common chord.

8. Show that the perspectivity that transforms the points on one circle into the inverse points on another is composed of a similitude which transforms the first circle into the second and an involution which leaves the second circle unchanged.

9. Show that the invariant (Art. 135) of a perspective similitude is positive or negative according as the similitude is direct or inverse.

10. The invariant of a perspective congruence is $+1$ or -1 according as the congruence is direct or inverse.

11. Show that the resultant of two perspectivities in the same plane which have a common center is a third perspectivity having the same center and whose axis is concurrent with the axes of the original perspectivities. Show also that the invariant of the resultant perspectivity is the product of the invariants of the original perspectivities.

147. Collineation in the Plane. Self-corresponding Elements.—

1. *When two collinearly related planes are superposed but are not in perspective position, they cannot have more than three self-corresponding points unless they coincide element to element.*

For suppose that four points A, B, C, D coincide with their corresponding points A_1, B_1, C_1, D_1. No three of these points can lie upon the same straight line, for then this line would be a self-corresponding point-row and the planes in perspective position, contrary to hypothesis. Hence the points A, B, C, D are the vertices of a complete quadrangle. The sheaf whose center is A coincides with the sheaf whose center is A_1, since the rays AB, AC, AD coincide with their corresponding rays. Similarly, the sheaves B and B_1 coincide. Any point P of the plane must coincide with its corresponding point P_1, since the rays joining P to A and B coincide with their corresponding rays. Therefore, the planes coincide element to element.

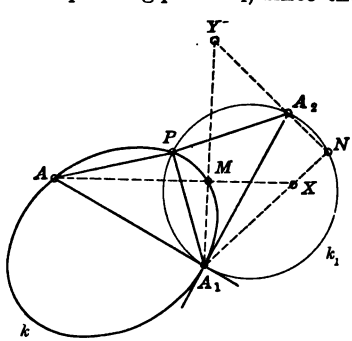


FIG. 109.

On the other hand, it is evident that a collineation can have three self-corresponding points. For, if we choose the vertices of a complete quadrangle A, B, C, D to correspond respectively to the vertices A, B, C, D_1 of another complete quadrangle, a collineation is completely and uniquely determined (Art. 131) which has the three points A, B, C as self-corresponding points.

If a collineation in a plane has three self-corresponding points forming a triangle ABC, the sides of this triangle are self-corresponding lines, but the point-rows along these lines do not coincide point to point. The point-rows along AB, for example, are projectively related and have A and B as self-corresponding points.

2. *Two superposed and collinearly related planes, not in perspective position nor coincident, must have at least one self-corresponding point and one self-corresponding line.*

Since the collineation is not a perspectivity, there are points in the plane which do not correspond to themselves nor lie on any self-corresponding line. Let A be such a point (Fig. 109). Suppose A corresponds to A_1 and, in turn, A_1 corresponds to A_2. The points A, A_1, A_2 cannot lie on the same line, for then this line would be a self-corresponding line.

The projectively related sheaves whose centers are A and A_1 generate a conic k, and the sheaves whose centers are A_1 and A_2, a second conic k_1. The conic k passes through A and A_1 and touches A_1A_2 at A_1 since the line A_1A_2 corresponds to AA_1. Likewise, the conic k_1 passes through A_1 and A_2 and touches AA_1 at A_1. The conics do not touch each other at A_1, since their tangents at this point do not coincide. Hence, the conic k crosses k_1 at A_1. The point A is necessarily outside k_1, and therefore k is partly outside and partly inside k_1, and must consequently cross k_1 in at least one other point besides A_1. This second point P, whose existence is thus demonstrated, is a self-corresponding point, since the ray AP corresponds to the ray A_1P, and, in turn, the ray A_1P corresponds to the ray A_2P.

The two conics k and k_1 serve to construct the point corresponding to any given point in the plane. Thus, if X is any point in the plane, the line AX corresponds to the line A_1Y, the two lines meeting in the point M of the conic k. Again, the line A_1X corresponds to the line A_2Y, the two lines meeting in the point N of the conic k_1. The point Y, corresponding to X, is the intersection of the lines A_1M and A_2N.

By means of the principle of duality, each step of the foregoing proof can be replaced by its reciprocal, and we thus see that a collineation in a plane must have at least one self-corresponding line.

The conics k and k_1 can meet in at most four points without coinciding. If these points are A_1, P, Q, R, then P, Q, R are self-corresponding points of the collineation, and the sides of the triangle PQR are self-corresponding lines.

Two of the points P, Q, R may coincide in a single point. The conics k and k_1 then touch each other at this point, and the collineation has two self-corresponding points and two self-corresponding lines. One of these lines joins the self-corresponding points, and the other passes through one of them and touches both conics there.

The results of the foregoing discussion are brought together in the following theorem.

3. *Two superposed and collinearly related planes, not in perspective position nor coincident, have either one point and one line as self-*

corresponding elements; or two points and two lines as self-corresponding elements, in which case one self-corresponding line joins the self-corresponding points and the other passes through one of them; or the vertices and sides of a triangle as self-corresponding elements.

Exercises

1. Given two conics k and k_1 in the same plane. Construct the polar lines $a, b, c,$—— of a series of points $A, B, C,$—— with respect to k and the poles $A_1, B_1, C_1,$—— of $a, b, c,$—— with respect to k_1. Show that the correspondence between the points $A, B, C,$—— and $A_1, B_1, C_1,$—— is a collineation having as self-corresponding elements the vertices and sides of the self-polar triangle common to k and k_1.

2. Draw two conics intersecting in four points. Choose one of these points for the point A_1 in Fig. 109 and construct a number of corresponding elements of the collineation determined by the conics. In particular, show that the remaining three points of intersection of the two conics are self-corresponding points of the collineation.

3. Prove that a collineation in a plane must have at least one self-corresponding line by means of the suggestion in Art. 147.

4. If in a plane, the vertices A, B, C, D of a complete quadrangle correspond respectively to the points A, B, C, D_1, show how to construct the point P_1 corresponding to any point P in the plane.

CHAPTER XV

POLARITIES IN A PLANE AND IN A BUNDLE

148. The Polarity in a Plane.—When two planes are reciprocally related and superposed they form a duality in their common plane (Art. 128). This duality does not, in general, correlate the elements of the plane so that any two corresponding elements, point and line, correspond to each other doubly. That is, if the superposed planes are α and α_1, and the line a of α corresponds to the point A_1 of α_1, it is not true, in general, that A_1 of α corresponds to a of α_1. If, however, the duality is such that every pair of corresponding elements is doubly corresponding, the duality is **involutoric,** and is called a **polarity.**

That polarities exist in a plane is evident, since the correspondence between pole and polar line with respect to a fixed conic is involutoric and forms a duality in the plane.

Two elements, point and line, that correspond to each other in a polarity are called **pole** and **polar line** respectively. Since the superposed planes constituting the polarity are projectively related, a pair of incident elements in one corresponds to a pair of incident elements in the other. Hence:

Of two points in a polarity, either each or neither lies upon the polar line of the other; and of two lines, either each or neither passes through the pole of the other.

Two points in a polarity are **conjugate points** if each lies on the polar line of the other, and two lines are **conjugate lines** if each passes through the pole of the other.

A point is **self-conjugate** if it lies on its own polar line, and a line is **self-conjugate** if it passes through its own pole.

To every figure in the plane corresponds a **polar-figure.** To a triangle corresponds a polar triangle; to a curve of second order, an envelope of second class; and so on.

A triangle is **self-polar** if each side is the polar line of the opposite vertex.

149. Construction of a Polarity in a Plane.—

A polarity is determined having given a self-polar triangle and a point, not on any of the sides of the triangle, together with its corresponding line, not passing through any of the vertices of the triangle.

It is evident that the given elements determine a duality in the plane. For, if A, B, C, and a, b, c are respectively the vertices and opposite sides of the given triangle, and D and d are the given point and its corresponding line (Fig. 110), then A, B, C, D form the vertices of a complete quadrangle and a, b, c, d,

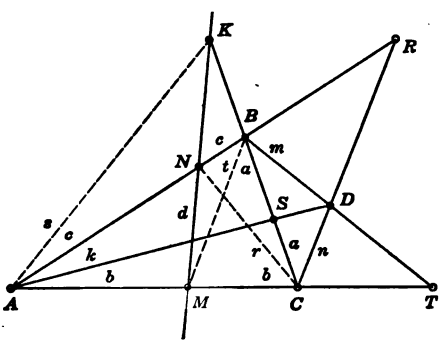


FIG. 110.

the sides of the corresponding complete quadrilateral (Art. 131).

We have now to show that this duality is involutoric and hence forms a polarity. Join D to the vertices A, B, C by the lines k, m, n respectively. To these lines correspond the points K, M, N where d meets the sides a, b, c. Let the lines k, m, n meet the sides a, b, c in the points S, T, R. To these points correspond the lines s, t, r joining A, B, C to K, M, N respectively. From the definition of projectivity we have,

$$SCKB \overline{\wedge} sckb \overline{\overline{\wedge}} KBSC$$

Therefore,

$$SCKB \overline{\wedge} KBSC$$

and we see that along the side a of the self-polar triangle there is an involution in which any point corresponds to the intersection of its corresponding line with a. Similarly, along the sides b and c exist involutions formed in like manner.

If p is any line in the plane meeting a, b, c in the points K', M', N', then these points correspond, in the involutions on a, b, c, to three points S', T', R', such that AS', BT', CR' meet in the point P corresponding to the line p. But the line corresponding to P must be p, since the lines AP, BP, CP meet the sides a, b, c in the points S', T', R' and these correspond, in the involutions on a, b, c, to K', M', N' respectively. Hence P and p correspond doubly and the duality is, therefore, a polarity.

150. Self-conjugate Points in a Polarity.—If the involution on a side of a self-polar triangle is hyperbolic, the foci of the in-

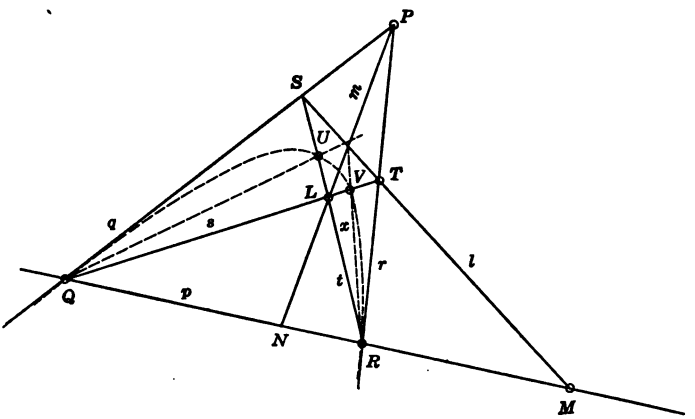


FIG. 111.

volution are self-conjugate points in the polarity, since a focus lies on its polar line.

1. *There is no line of a polarity all of whose points are self-conjugate; and no point all of whose rays are self-conjugate.*

For, suppose p is such a line (Fig. 111). Let Q and R be any two points on it, and let P be its pole. The polar lines of Q and R are the lines PQ and PR respectively. Any point on PQ, as S, has for its polar line a line s passing through Q and meeting PR in T. The polar line of T joins S and R, meeting QT in L. The polar line of L is ST, meeting p in M. Now the polar line of M is PL which cannot pass through M, since M is the intersec-

tion of two diagonals of the complete quadrangle $LSPT$ and PL is the third diagonal. Hence, M is not a self-conjugate point, contrary to the supposition made. We conclude, therefore, that no line can have all its points self-conjugate.

The principle of duality enables us to say that, reciprocally, there is no point in the plane all of whose rays are self-conjugate lines.

2. *In a given polarity in a plane there are an infinity of self-polar triangles.*

For, let p be any line in the plane and P its pole (Fig. 112). Upon p choose any point Q which is not a self-conjugate point (*cf.* 1). The polar line of Q passes through P and meets p in a point R distinct from Q. The polar line of R is then PQ, and PQR is a self-polar triangle. Hence a self-polar triangle can always be constructed having any line of the polarity as one side. Indeed an infinity of such triangles can be constructed having p as one side, since Q can be any point on p which is not self-conjugate.

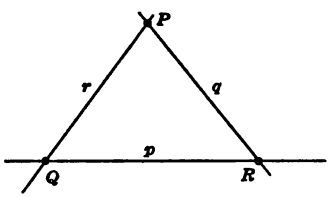


FIG. 112.

It follows from this proposition and Art. 149 that:

3. *Along any line p of a polarity in a plane there exists an involution in which any point corresponds to the intersection of its polar line with p; that is, to its conjugate point on p.*

If this involution is hyperbolic, its foci are self-conjugate points of the polarity. Hence:

4. *Not more than two self-conjugate points of a polarity can lie on any line in the plane.*

If a line p is self-conjugate, its pole P is the only self-conjugate point on it, since the polar lines of all the other points of it must pass through P. The involution on a self-conjugate line is therefore parabolic or trivial (Art. 106).

5. *Every self-conjugate point S of a polarity is one focus of the involutions existing on the lines passing through S.*

6. *The locus of all the self-conjugate points of a polarity is a conic.*

For no line in the plane can meet this locus in more than two

points. These points are the foci of the involution on the line and are separated harmonically by any pair of points in the involution (Art. 104, 2). Thus, in Fig. 111, V is a self-conjugate point of the polarity on the line s, since it is separated harmonically from Q by the conjugate points L and T. Similarly, U is a self-conjugate point on t. If x is the line RV, the conic generated by the sheaves

$$qsp—\overline{\wedge} pxr——$$

passes through V and U and is tangent to q and r at the points Q and R. This conic is the locus of self-conjugate points of the polarity. The triangle LST is self-polar with respect to this conic, and the entire polarity appears as the correspondence between pole and polar line with respect to this conic.

The locus of the self-conjugate points of a polarity can never consist of two straight lines by virtue of proposition 1.

The tangents to the locus of self-conjugate points are self-conjugate lines of the polarity, since each tangent contains but one self-conjugate point.

151. Classification of Polarities in a Plane.—

In a given polarity in a plane, the involutions on the sides of any self-polar triangle are either all elliptic or else one of them is elliptic and the other two are hyperbolic.

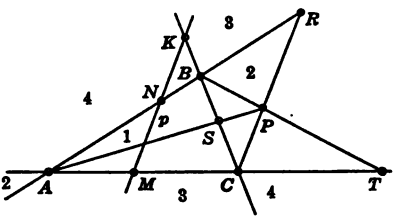


FIG. 113.

Suppose the polarity is determined by the self-polar triangle ABC and a pair of corresponding elements p, P (Fig. 113). The triangle ABC divides the plane into four regions each of which is bounded by three segments. For example, the region 2 is bounded by the finite segment BC, the infinite segment CA, and the infinite segment AB. The point P must occupy one of these regions which we will call the region P. The polar line p cannot

pass through any vertex nor meet all three of the segments bounding the region P. If p meets one of these segments it must also meet another, since if it enters the region P it must pass out of it. There are then two cases to consider.

First, p does not meet any of the segments bounding the region P. This is the case shown in the figure.

Second, p meets two of the segments bounding the region P.

If p meets the sides of the triangles a, b, c in the points K, M, N, respectively, the polar lines of these points are the lines AP, BP, CP meeting the sides a, b, c in the points S, T, R, conjugate respectively to K, M, N. The points S, T, R are necessarily on the segments bounding the region P.

In the first case, the points K, M, N are none of them on the segments bounding the region P, and hence the involution on any side of the triangle is determined by two pairs of points which separate each other. For example, the involution on the side a is determined by the pairs B, C and K, S which separate each other. Hence all three of the involutions are elliptic. The involution along any line of the plane is necessarily elliptic, since no line can enter the region containing its pole. There are, therefore, no self-conjugate points or self-conjugate lines in the plane. A polarity of this type is called **uniform.**

In the second case, two of the three points K, M, N are on segments bounding the region P while the third is not. Hence on two of the sides a, b, c, the involutions are determined by pairs of points which do not separate each other while on the third side, the involution is determined by pairs which do separate each other. Consequently, in this case, two involutions are hyperbolic while the third is elliptic. There are self-conjugate points in the plane whose locus is a conic. Every self-polar triangle in the polarity is self-polar with respect to the locus of self-conjugate points, and consequently two of the involutions on its sides are hyperbolic and one is elliptic. A polarity of this type is called **non-uniform.**

A non-uniform polarity defines the conic which is the locus of its self-conjugate points, since this conic is uniquely determined by the polarity. Any line in the plane meets this conic in two real points, in one real point, or in conjugate imaginary points according as the involution along the line is hyperbolic, parabolic, or elliptic.

A uniform polarity in a plane defines an **imaginary conic** in the plane. Every line of the plane meets this conic in two imaginary points defined by the elliptic involutions along the line.

Exercises

1. Show that in a polarity there is no sheaf of lines all of whose rays are self-conjugate (*cf.* Art. 150, 1).

2. Show that about any point P of a polarity there is an involution in which any ray corresponds to its conjugate ray through P.

3. If P is a self-conjugate point of a polarity, its polar line is the only self-conjugate line through it.

4. In a uniform polarity, the involutions about the vertices of any self-polar triangle are all elliptic; in a non-uniform polarity, two of these involutions are hyperbolic and the third is elliptic.

5. Show that a polarity is determined having given a self-polar triangle, the involution on one of its sides, and a pair of conjugate points not lying on the sides of the triangle.

6. Show that a polarity is determined by choosing the vertices of any simple pentagon to correspond respectively to the opposite sides.

7. Show that a polarity is determined having given the involutions along two non-conjugate lines a, b and the pole of one of these lines. This pole must lie on the line joining the points corresponding to the point $(a\ b)$ in the given involutions.

8. Show that a polarity is determined having given the involutions along two conjugate lines.

9. If two pairs of opposite sides of a complete quadrangle are pairs of conjugate lines in a polarity, show that the third pair is a pair of conjugate lines in the same polarity. (Theorem of Von Staudt-Hesse.)

152. The Polarity in a Bundle.—If we project a polarity in a plane from a point not lying in the plane, we obtain a polarity in the bundle whose center is the point from which the projection is made. This polarity is uniform or non-uniform according as it is the projector of a uniform or a non-uniform polarity.

A polarity in a bundle is an involutoric duality between the lines and planes of the bundle. The plane corresponding to any line is the **polar plane** of the line; and the line corresponding to any plane is the **pole-ray** of the plane.

Two lines, each lying in the polar plane of the other, are **conjugate lines**; two planes, each containing the pole-ray of the other, are **conjugate planes.**

A line lying in its own polar plane is **self-conjugate**; a plane passing through its own pole-ray is **self-conjugate.**

All self-conjugate lines of a non-uniform polarity in a bundle lie upon a cone of second order, and all self-conjugate planes are tangent planes to this cone.

A uniform polarity in a bundle has no real self-conjugate lines or self-conjugate planes. It defines an **imaginary cone** in space.

153. The Orthogonal Polarity.—If two orthogonally correlated bundles (Art. 125) are superposed, that is, if they are concentric, the resulting duality in their common bundle is a polarity. For, if a is any line of the bundle, its corresponding plane α is perpendicular to it. The line corresponding, in turn, to α is perpendicular to α and is, therefore, the line a. Any two lines of the bundle, as a and b, determine a plane γ. The polar planes of a and b are perpendicular to a and b respectively, and intersect in a line c which is the pole-ray of γ and is also perpendicular to γ. Hence γ is the polar plane of c. The duality in the bundle is, therefore, involutoric and thus forms a polarity. This polarity is called the **orthogonal polarity** in the bundle.

An orthogonal polarity is necessarily uniform, since no line ever lies in its polar plane.

In each plane of the bundle pairs of conjugate lines are so situated that each line of the pair is perpendicular to the other. The involution of conjugate lines in any plane is, therefore, circular (Art. 108). Also, about any line of the bundle pairs of conjugate planes are so situated that each plane of the pair is perpendicular to the other. Thus, the involution of conjugate planes about any line of the bundle may be called a circular involution.

Any tri-rectangular pyramid, whose vertex is the center of the bundle, is **self-polar,** since the edges are the pole-rays of the opposite faces. Conversely, any self-polar pyramid is tri-rectangular.

154. Polarity and Anti-polarity with Respect to a Circle.—A given circle establishes a non-uniform polarity in its plane in which two corresponding elements, point and line, are pole and polar line with respect to the circle. Let C be the center of the

circle and r, the length of the radius (Fig. 114). If A is any point in the plan, its polar line a' meets the diameter through A in a point M' such that

$$\frac{AC}{r} = \frac{r}{CM'}, \text{ or } CM' = \frac{r^2}{AC}.$$

If we cut an orthogonal polarity by any plane not passing through its vertex O, we obtain a uniform polarity in the plane.

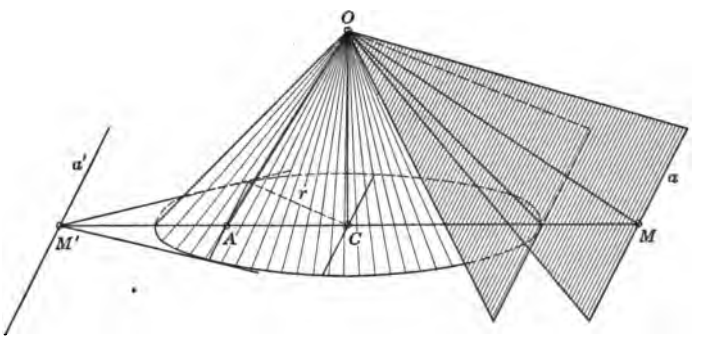


FIG. 114.

Let OC be the perpendicular from the center of the bundle upon the cutting plane. With OC as radius and C as center, describe a circle in the plane. The lines of the bundle drawn to points on this circle are rays of a right circular cone whose vertical angle is a right angle, hence the polar plane of any one of these rays is tangent to the cone along a line diametrically opposite to the ray. The polarity in the plane, then, is such that the polar line of any point D on the circle is the tangent at the opposite end of the diameter through D. The polar line of any point as A is a line a perpendicular to the diameter AC at a point M such that

$$AC \cdot CM = CO^2 = r^2.$$

It follows that $CM = CM'$; the two polar lines a and a' cut the diameter AC at points equidistant from the center of the circle and on opposite sides of it. On account of this property, any section of the orthogonal polarity in a bundle is called an **anti-polarity** with respect to a circle, the radius of the circle being equal

in length to the perpendicular from the center of the bundle upon the cutting plane.

In an anti-polarity with respect to a circle, the polar line of the center of the circle is the ideal line, the involution of conjugate lines about the center is circular, and the involution of conjugate points on the ideal line coincides with the involution determined upon this line by the circle.

155. The Absolute Polarity.—*The sections of all the orthogonal polarities in space by the ideal plane coincide in one and the same polarity in the ideal plane.*

For, suppose p and p' are parallel lines in two orthogonal polarities. The polar planes, π and π' are necessarily parallel to each other; hence the two pairs of elements p, π and p', π' are cut by the ideal plane in the same pair of elements, and these are pole and polar line in the polarity in the ideal plane. This polarity is called the **absolute polarity.** It is uniform and defines the **imaginary circle at infinity.**

Exercises

1. Show how to construct a uniform polarity in a plane.
2. Construct five points of the conic which is the locus of self-conjugate points in a given non-uniform polarity.
3. The section of an orthogonal polarity, by a plane passing through the center of the bundle, consists of pairs of conjugate rays forming a circular involution.
4. In a given polarity in a plane, show that two polar triangles having no elements in common are in perspective position. Use Art. 151, exercise 9.
5. A conic can be circumscribed about two polar triangles in a plane. The sides of the triangles touch another conic.

156. Double Polarities in a Plane and in a Bundle.—If two polarities exist together in a plane or in a bundle, we shall speak of them as forming a **double polarity** in the plane or in the bundle.

Let P_1 and P_2 denote two polarities forming a double polarity in a plane. In general, any line a in the plane has two poles A_1 and A_2, one in each polarity; a triangle whose sides are a, b, c, has two polar triangles A_1, B_1, C_1 and A_2, B_2, C_2, one in each polarity; a sheaf of rays has two point-rows of poles which are projectively related to each other, since each is projectively related to the sheaf

of rays. Any point A in the plane has, in general, two polar lines a_1 and a_2, one in each polarity; a point-row has two sheaves of polar lines which are projectively related to each other, since each is projectively related to the point-row. It follows that there is a **collineation** in the plane in which two corresponding points are the poles of the same line, and two corresponding lines are the polar lines of the same point.

If we regard a polarity as an operation which transforms all the elements of a plane into their respective polar elements, we can say that the collineation, whose existence has just been shown, is the result of performing the operations P_1 and P_2, one after the other. For, if A_1 and A_2 are the poles of a in P_1 and P_2, respectively, then P_1 changes A_1 into a, and P_2 changes a into A_2. Symbolically expressed,

$$P_1(A_1) = a,$$
$$P_2(a) = A_2,$$

and hence,

$$P_2P_1(A_1) = P_2(a) = A_2.$$

For convenience, we shall denote the resultant collineation of the two polarities by the letter R. We can then write,

$$P_2P_1 = R.$$

In the same way, two polarities in a bundle have a resultant collineation in which any two corresponding planes are the polar planes of the same ray, and any two corresponding rays are the pole-rays of the same plane. We shall also denote this collineation by the letter R.

157. Common Elements in a Double Polarity.—A point and a line are **common elements** in a double polarity in a plane if they are pole and polar line in both polarities; a ray and a plane are **common elements** in a double polarity in a bundle if they are pole-ray and polar plane in both polarities.

The common elements in a double polarity (P_2, P_1) are the self-corresponding elements of the resultant collineation R, and conversely.

For, if $P_1(A) = a$ and $P_2(a) = A$, then $P_2P_1(A) = P_2(a) = A$, or $R(A) = A$. Similarly, $R(a) = a$. Thus the common elements, A, a, in (P_2, P_1) are self-corresponding elements in R.

Conversely, if $P_2P_1(A) = A$ and $P_1(A) = a$, then $P_2(a) = A$,

$P_1(a) = A$, and $P_2(A) = a$. Thus self-corresponding elements of R are common elements in (P_2, P_1).

There are two cases to consider:

1. If R is a perspectivity, then, in a plane, (P_2, P_1) has a common point-row of poles along the axis of perspectivity and a common sheaf of polar lines about the center of perspectivity. The center and the axis are pole and polar line in each polarity.

As an example, the double polarity formed by the poles and polar lines with respect to two concentric circles in a plane has a resultant collineation which is a perspectivity. The axis of this perspectivity is the ideal line, and the center is the common center of the two circles.

In a bundle, (P_2, P_1) has a common sheaf of pole-rays and a common sheaf of polar planes. The axis of the sheaf of planes and the plane of the sheaf of rays are pole-ray and polar plane in each polarity (Art. 133).

2. If R is not a perspectivity, then (P_2, P_1) in a plane has at least one common point and one common line which are pole and polar line in both polarities, or at most the vertices and sides of a common self-polar triangle. For example, the double polarity formed by the poles and polar lines with respect to two conics, having four common points, has the vertices and sides of the common self-polar triangle as common elements.

In a bundle, (P_2, P_1) has at least one common ray and one common plane which are pole-ray and polar plane in both polarities, or at most the edges and faces of a common self-polar pyramid (Art. 147, 3).

158. Double Conjugate Elements.—In a double polarity in a plane, any point as A (Fig. 115) has, in general, two polar lines a_1 and a_2 which intersect in a point B. The polar lines of B intersect in A. The points A and B are conjugate to each other in both polarities and shall be called **double conjugate points.** Reciprocally, two lines a and b are **double conjugate lines,** if they are conjugate lines in both polarities, that is, if each joins the poles of the other.

In a bundle, two rays are **double conjugate rays,** if they are conjugate rays in both polarities; and two planes are **double conjugate planes,** if they are conjugate planes in both polarities.

If, in Fig. 115, A describes the point-row s, B will, in general, describe a conic k. For the polar lines of A will describe sheaves about the poles S_1 and S_2 of s which are projectively related to each other, since each is projectively related to the point-row described by A. These sheaves generate the conic k. The line s and the conic k correspond to each other point to point. Each is the **locus of double conjugates** to the points of the other.

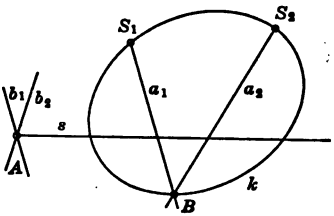


FIG. 115.

If, however, s passes through a common point P of the double polarity (Fig. 116), the sheaves S_1 and S_2 are in perspective position, since the polar lines of P coincide in the line $S_1 S_2$. The locus of double conjugates to the points of s is then a line t which also passes through P, since the polar lines of R pass through P. The lines s and t are corresponding loci of double conjugate points.

Reciprocally, if a and b are double conjugate lines in a plane and a describes a sheaf of rays about a point S, b, will, in general,

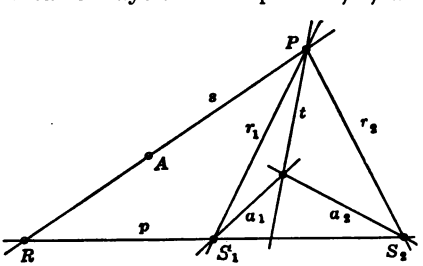


FIG. 116.

describe an envelope of second class K whose rays join corresponding poles on the polar lines s_1 and s_2 of S. But if S lies on a common line of the double polarity, the two point-rows of poles corresponding to rays of S are in perspective position, and b will then describe a sheaf of rays of first class about a point T also on the same common line.

Similar conclusions hold concerning corresponding loci of double conjugate elements in a bundle.

159. Confocal Elements in a Double Polarity.—When two polarities are in the same plane, there are two involutions of conjugate points along each line of the plane, and two involutions of conjugate rays about each point of the plane. Any line upon which the two involutions coincide is a **confocal line** of the double polarity, and any point about which the two involutions coincide is a **confocal point** of the double polarity.

Similarly, when two polarities are in the same bundle, there are two involutions of conjugate rays in each plane of the bundle, and two involutions of conjugate planes about each ray of the bundle. Any plane in which the two involutions coincide is a **confocal plane** of the double polarity, and any line about which the two involutions coincide is a **confocal axis** of the double polarity.

160. Construction of the Confocal Elements of a Double Polarity.—

1. If the resultant collineation $P_2P_1 = R$ is a perspectivity, the polar lines of any point A of its axis coincide and intersect the axis in the double conjugate point B (Fig. 117). Reciprocally, the poles of any line a through the center coincide and determine the double conjugate ray b. Consequently, the involutions determined by P_1 and P_2 along the axis and about the center coincide, element to element. Hence the axis of R is a confocal line and the center of R is a confocal point. It is easy to see that these are the only confocal elements in the plane.

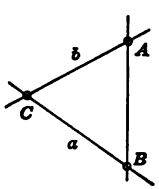


FIG. 117.

In a bundle, the involutions about the axis of the common sheaf of planes coincide, and the involutions in the plane of the common sheaf of rays coincide. These are the only confocal elements in the bundle.

2. If R is not a perspectivity, we shall suppose that P_1 and P_2 have a common self-polar triangle ABC (Fig. 118). Any line through a vertex of this triangle has for its corresponding locus of double conjugate points a line through the same vertex (Art. 158). Thus, if S and T are double conjugate points, the lines joining S and T to the vertices A, B, C are pairs of corresponding loci of double conjugate points. If, for example, S describes the line l, T described the line n, and conversely. The two point-

rows, l and n, are projectively related and are projected from B, say, in projectively related sheaves of rays described respectively by the rays s and t. These sheaves are not only projectively related, but are in involution, since a pair of rays is a pair of corresponding loci of double conjugate points. Hence:

About each vertex of the self-polar triangle there is an involution such that to any ray corresponds its locus of double conjugate points.

The three involutions will be all hyperbolic if any pair of double conjugate points, as S and T, occupy the same region of the plane

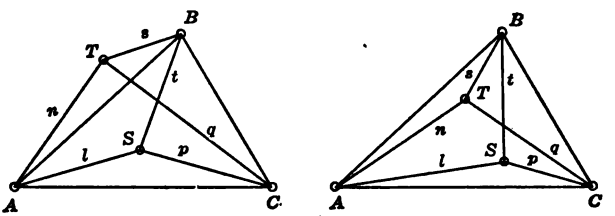


FIG. 118.

defined by the self-polar triangle ABC. If S and T occupy different regions, two of the involutions are elliptic and one is hyperbolic.

The focal rays in any one of these involutions are lines each of which is its own locus of double conjugate points, that is, they are confocal lines. We conclude therefore:

There are either six real confocal lines intersecting in pairs in the vertices of the common self-polar triangle, or else there are two real confocal lines meeting in one vertex of the common self-polar triangle.

The involutions which thus determine the confocal lines are known as soon as one pair of double conjugate points is known. For two sides of the self-polar triangle form a pair of rays in one of these involutions, since the polar lines of a point on any side intersect in the opposite vertex. The involution about the vertex B, for example, is determined by the pairs TB, SB and AB, CB.

Reciprocally we have the following statements:

On each side of the common self-polar triangle there is an involution such that to any point S corresponds the center T of its sheaf of double

conjugate rays. The two vertices of the triangle on the same side are corresponding points in the involution on that side. One of these involutions is always hyperbolic; the other two may be hyperbolic or elliptic. There are, therefore, either six real confocal points lying by pairs on the sides of the self-polar triangle, or else there are two real confocal points lying on one side of the self-polar triangle.

We see at once by projection that, if a double polarity in a bundle has a common self-polar pyramid:

A double polarity in a bundle has either six real confocal planes intersecting by pairs in the edges of the common self-polar pyramid, or else it has two real confocal planes meeting in one edge of the common self-polar pyramid; and it has either six real confocal axes lying by pairs in the faces of the common self-polar pyramid, or else it has two real confocal axes lying in one face of the common self-polar pyramid.

Exercises

1. A double polarity in a plane is determined by two real conics which intersect in four real points A, B, C, D.

(a) The polarities have a real common self-polar triangle which is the diagonal triangle of the quadrangle $ABCD$.

(b) The six sides of the quadrangle $ABCD$ are confocal lines.

(c) The six vertices of the quadrilateral whose sides are the common tangents to the conics are confocal points.

2. A double polarity is determined by two confocal conics.

(a) The common self-polar triangle consists of the axes and the ideal line.

(b) The foci are the real confocal points. Common chards are real confocal lines.

3. A given circle determines a polarity and an anti-polarity in its plane. Show that the resultant collineation of the double polarity in the plane is a perspectivity.

4. A double polarity is determined by two circles in a plane which are not concentric.

(a) Construct the common self-polar triangle.

(b) The centers of similitude are confocal points. Are there more than two real confocal points?

(c) Determine the number and position of the real confocal lines.

5. A circle is concentric with an ellipse. Show that the anti-polar-

ity with respect to the circle and the polarity with respect to the ellipse have in common a real self-polar triangle and that there are two real confocal lines and two real confocal points.

161. Application to Cones of the Second Order.—A cone of second order sets up a polarity in the bundle to which it belongs such that corresponding elements are pole-ray and polar plane with respect to the cone. This polarity, which we shall denote by P_1, is non-uniform; its self-conjugate lines are rays of the cone and its self-conjugate planes are tangent planes to the cone.

We shall now consider the orthogonal polarity P_2 in the same bundle. The double polarity (P_2, P_1) has a resultant collineation R; and there are two cases to consider according as R is a perspectivity or not a perspectivity.

1. If R is a perspectivity, then P_1 and P_2 have in common a sheaf of rays and a sheaf of planes (Art. 157, 1) such that the plane α of the sheaf of rays and the axis a of the sheaf of planes are polar plane and pole-ray in both polarities; that is, the plane α is perpendicular to the line a, since P_2 is the orthogonal polarity (Fig. 119). Moreover, α is the confocal plane and a is the confocal axis, and the coinciding involutions are circular (Art. 160, 1). Consequently, if we cut the cone by plane parallel to α, and which cuts the line a in the point A, we obtain a conic such that the involution of conjugate rays about A is circular. Hence, A is a focus of the conic. But since the directrix (the intersection of α with the plane of the conic) is infinitely distant, the conic is a circle. Hence, the cone is a right circular cone whose axis is a.

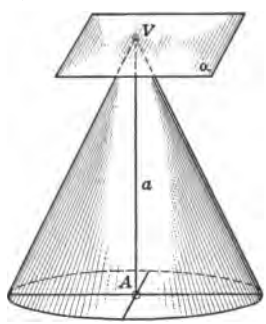


FIG. 119.

2. *If R is not a perspectivity, P_1 and P_2 have in common a real self-polar pyramid.*

For the two polarities have in common at least one line a and one plane α which are pole-ray and polar plane in both polarities. Hence a and α are perpendicular to each other. In the plane α there are two involutions of conjugate rays, one for each

polarity. The involution belonging to P_2 is necessarily elliptic, and hence the two involutions have two real rays in common (Art. 117, 3) which we will call b and c. These rays, being con-

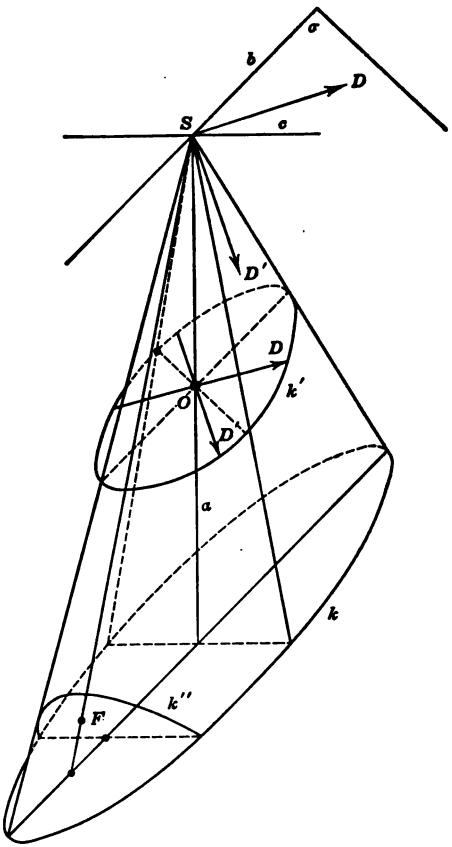


FIG. 120.

jugate in the polarity P_2, are perpendicular to each other and each is perpendicular to a, since they lie in the plane α. The three rays a, b, c form the edges of the common self-polar pyramid and are the **principal axes** of the cone. The faces of the self-polar pyramid are the **diametral planes** of the cone.

One of the axes must be within the cone, since of the involutions about the axes which belong to P_1, one is necessarily elliptic (Art. 151, second case). This is the **major axis** of the cone and is denoted by the letter a (Fig. 120). Any section of the cone by a plane perpendicular to a is an ellipse whose axes are parallel to the other two axes of the cone. That axis of the cone which is parallel to the major axis of the ellipse is the **mean axis** of the cone and is denoted by the letter b. The third axis is the **minor axis** of the cone and is denoted by the letter c. Any section of the cone perpendicular to either b or c is a hyperbola.

162. Cyclic Planes and Focal Axes of Cones.—*If the resultant collineation $R = P_2P_1$ is not a perspectivity, there are two real confocal planes intersecting in the mean axis of the cone and two real confocal axes in the diametral plane perpendicular to the minor axis of the cone.*

For, if we cut the double polarity by a plane perpendicular to the major axis of the cone, we obtain a double polarity in the cutting plane composed of a polarity with respect to the ellipse (section of the cone) and an anti-polarity with respect to a circle concentric with the ellipse (Art. 154). This double polarity has a real self-polar triangle whose sides are the axes of the ellipse and the ideal line. There are but two real confocal lines, and these are parallel to the major axis of the ellipse. Also there are but two real confocal points, and these lie on the major axis of the ellipse (cf. Art. 160, exercise 5). These confocal elements are the traces of the confocal elements of the double polarity in the bundle.

The confocal planes of the double polarity are the **cyclic planes** of the cone; and the confocal axes of the double polarity are the **focal axes of the cone.**

Any section of a cone parallel to a cyclic plane is a circle.

For, if σ is a cyclic plane and SD is any ray in σ through S, the conjugate ray SD' is perpendicular to SD, since the coinciding involutions in σ are circular. The polar planes of SD and of SD' in P_1 are cut by any plane parallel to σ in a pair of conjugate diameters of the conic k', cut from the cone (Fig. 120). It follows that any two conjugate diameters of k' are mutually perpendicular, and hence k' is a circle.

Any plane perpendicular to a focal axis of a cone cuts the cone

in a conic k'' and the focal axis in a point F which is one focus of k''.

For the coinciding involutions about the focal axis are circular. Conjugate planes in P_1 are, therefore, cut in normal conjugate rays with respect to the conic k''. F is therefore a focus of k''.

163. Quadric Transformations.—When two polarities exist in the same plane, the transformation which changes every point of the plane into its double conjugate point is called a **quadric transformation.** We have already noticed the particular quadric transformation that arises from non-uniform polarities in the same plane (Art. 72, exercise 6).

A quadric transformation is not a collineation, since any point-row s corresponds, in general, to a conic k which is the locus of double conjugates to the points of s (Art. 158).

The nature of a quadric transformation depends upon the two polarities from which it arises. If P_1 and P_2 are two polarities and R, their resultant collineation, is a perspectivity, then the double conjugate of any point in the plane is on the axis of perspectivity, since a pair of corresponding polar lines must meet on the axis. In this case the quadric transformation transforms every line in the plane into the axis of perspectivity, and is, for this reason, called degenerate or trivial.

If R is not a perspectivity, the self-corresponding elements of R are called the **fundamental elements** of the quadric transformation. Thus, if P_1 and P_2 have in common the vertices and sides of a self-polar triangle, then the vertices of this triangle are the **fundamental points,** and the sides are the **fundamental lines,** of the quadric transformation arising from the double polarity (P_2, P_1). A vertex of this triangle corresponds to any point on the opposite side, since the polar lines of a vertex coincide in the opposite side; and any point on a side of this triangle corresponds to the opposite vertex, since the polar lines of the point intersect in the opposite vertex.

1. *Any line s, not passing through any fundamental point of a quadric transformation, corresponds to a conic k containing all the fundamental points.*

For the line s meets the fundamental lines of the quadric transformation in points whose double conjugates are the fundamental

points of the quadric transformation. Therefore, the conic k passes through all the fundamental points.

2. *A line s passing through a fundamental point of a quadric transformation corresponds to a line t passing through the same fundamental point.*

For the locus of double conjugates to the points of s is a line t through the same fundamental point (Art. 158).

3. *A conic k passing through two of the fundamental points of a quadric transformation corresponds to a conic k' passing through the same two fundamental points.*

Let A and B be two fundamental points and P, any point of the conic k (Fig. 121). The lines s and s', joining P to A and B, correspond in the quadric transformation to lines t and t' intersecting in a point Q. P and Q are corresponding points in the quadric transformation. As P described the conic k, the lines s and s' describe projectively related sheaves of rays about A and B. But the sheaf described by s is projectively related to the sheaf described by t, and the sheaf described by s' is projectively related to the sheaf described by t' (Art. 160, 2). Hence, the sheaves described by t and t' are projectively related and generate a conic k', the locus of Q.

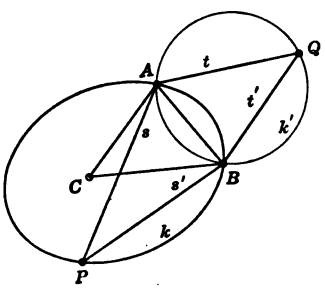


FIG. 121.

164. Perspective Quadric Transformations.—Let A be the center of a sheaf of rays of first class and P_1, a polarity in the same plane with the sheaf of rays.

The transformation which changes any point P into the point Q, the intersection of PA with the polar line of P, is a quadric transformation.

For, if P describes a point-row s (Fig. 122), the line PA describes the sheaf A and the polar line of P describes the sheaf of polar lines about the pole S of s. Hence,

$$\text{sheaf } A \overline{\overline{\wedge}} \text{ point-row } s \overline{\wedge} \text{ sheaf } S,$$

and therefore,

$$\text{sheaf } A \overline{\wedge} \text{ sheaf } S.$$

The intersection of corresponding rays in these sheaves describes a conic k, the locus of Q. Hence, in general, a point-row s corresponds to a conic k.

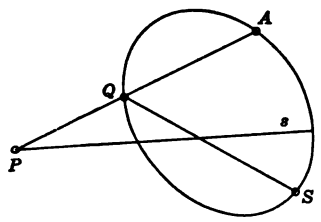


FIG. 122.

The transformation which changes P into Q is involutoric, since the polar line of Q passes through P.

Any line passing through A corresponds to itself but not point to point. A is a fundamental point of the quadric transformation. The polar line of A in P_1 is a fundamental line of the quadric transformation, since the polar line of any point on this line must pass through A.

If the polarity P_1 is non-uniform, it has a conic of self-conjugate points k'. Any point on k' is a self-corresponding point of the quadric transformation, since it lies on its polar line. Two corresponding points of the quadric transformation are collinear with A and harmonically separated by k'. If P_1 is uniform, there are no real self-corresponding points of the quadric transformation.

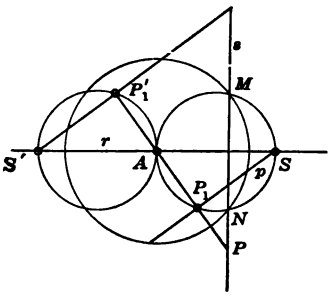


FIG. 123.

165. Inversion with Respect to a Circle.—Two important perspective quadric transformations are the inversions with respect to a fixed circle. When the polarity P_1 is either a polarity or an antipolarity with respect to a circle whose center is also the center of the sheaf A, the perspective quadric transformation is called an **inversion** with respect to the circle.

Suppose that P_1 is a polarity with respect to a fixed circle whose radius is r and whose center is A (Fig. 123). Let s be any line not passing through A. As P describes s, its polar line describes the

projectively related sheaf of lines about the pole S, and the line AP describes the sheaf A in perspective position with s. The sheaves S and A generate a circle on SA as diameter. This circle corresponds to s in the inversion. P and P_1 are corresponding points in the inversion.

Since P_1 is on the polar line of P with respect to the fixed circle, we have (Art. 154),

$$AP \cdot AP_1 = r^2.$$

On account of this relation, an inversion is often called a **transformation by reciprocal radii.**

If P_1 is an anti-polarity with respect to the fixed circle, the point P corresponds to the point P'_1, the intersection of AP with the anti-polar line of P. The line s corresponds to the circle on $S'A$ as diameter. In this case we have,

$$AP \cdot AP'_1 = -r^2,$$

the negative sign being used because AP and AP'_1 are measured in opposite directions.

In either of the inversions, the center of the fixed circle is called the **center of inversion.** We may distinguish between the two inversions by calling the one **positive,** and the other **negative.** The center of inversion is the fundamental point, and the ideal line is the fundamental line, of either inversion. Every point of the fixed circle is a self-corresponding point of a positive inversion. There are no self-corresponding points of a negative inversion.

166. Properties of Positive Inversions.—

1. *To a circle not passing through the center of inversion corresponds a circle not passing through the center of inversion.*

For, let O be the center of any circle k not passing through A (Fig. 124), and let AO meet k in S and T. Let P be any point on k, and construct the points corresponding to S, T, P in the inversion. Let these be the points S_1, T_1, P_1. Then we have,

$$AS \cdot AS_1 = AP \cdot AP_1 = r^2, \text{ or,}$$

$$AS_1 : AP_1 :: AP : AS.$$

Hence the triangles AS_1P_1 and APS are similar. Likewise, the triangles AT_1P_1 and APT are similar. It follows from elementary geometry that,

$$\angle S_1P_1T_1 = \angle SPT.$$

Therefore, as P describes the circle k, P_1 describes a circle k_1 on S_1T_1 as diameter. If k meets the fixed circle in real points, k_1 must pass through these points.

2. *The center of inversion is a center of similitude of any pair of corresponding circles k and k_1. Corresponding points on these circles are inverse points with respect to this center of similitude* (*Art.* 145).

3. *Any circle passing through a pair of corresponding points is inverse to itself, that is, corresponds to itself in the inversion.*

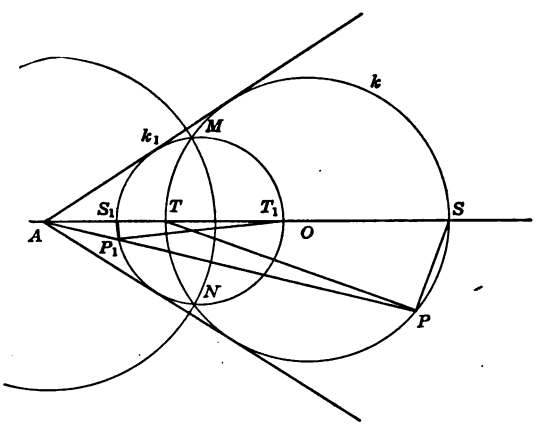


FIG. 124.

4. *An inversion transforms every point outside the fixed circle into a point inside the fixed circle, and vice versa.*

5. *Any two straight lines of the plane intersect in the same angle as do their corresponding circles.*

For, if we draw lines through the center of inversion and parallel to the given lines, these will touch the corresponding circles at the center of inversion. Consequently, corresponding angles of two inverse figures are equal.

Any transformation which does not change the magnitude of the angles of a figure is called a **conform transformation.** The inversions are conform transformations.

Exercises

1. A perspective quadric transformation arises from a polarity with respect to a fixed conic and a given sheaf of lines A. Show that the conic corresponding to any line not passing through A meets the fixed conic in the points where it is met by the polar line of A.

2. In an inversion, a system of concentric circles corresponds to a system of circles with a common radical axis.

3. If two circles touch each other, their inverse circles also touch each other.

4. Show that a circle inverse to itself cuts the fixed circle orthogonally.

5. Show how to invert two given circles into two equal circles.

6. Do the properties given in Art. 166 hold for negative inversions?

167. Circular Transformations.—If a plane is first transformed into itself by a similitude S and then inverted with respect to a fixed circle by a positive inversion I, the resultant transformation,

$$C = I \cdot S,$$

is called a **circular transformation.**

Any line in the plane can be regarded as a circle of infinite radius. With this proviso, we can say that:

1. *A circular transformation always changes circles into circles.* For the similitude S changes circles into circles (Art. 144, 2) and the inversion I also changes circles into circles.

2. *Circular transformations are conform.* For neither the similitude nor the inversion alters the angles of a figure.

Quadric transformations arising from two polarities in a plane and perspective quadric transformations have in common two important properties. First, they are one to one "point-transformations"; that is, to any point corresponds, in general, but a single point. The exceptions are the fundamental points. Any transformation having this property is called **birational.** Second, they are involutoric in character; that is, the points of any pair correspond to each other doubly.

A circular transformation is birational but is not, in general, involutoric.

3. *Any birational transformation not a similitude which changes circles into circles is a circular transformation or an inversion.*

Let C be the transformation in question. In the first place, C

must have a fundamental point A through which pass all the circles corresponding to the straight lines of the plane. For not all the lines of the plane can be changed into lines, since C is not a similitude. Consider two lines a and b; their corresponding circles meet in two points, only one of which can correspond to the intersection of a and b, since C is birational. The other must be a fundamental point A.

If, now, we invert the plane with A as the center of inversion, the resultant transformation, IC, must be a similitude, since it transforms lines into lines and circles into circles. Consequently,

$$IC = S.$$

If we invert again with the same inversion we have,

$$IIC = IS.$$

But the same inversion, performed twice in succession, does not alter the plane, since an inversion is involutoric. Hence,

$$C = IS,$$

and we conclude that C is a circular transformation. If the similitude S is the identity, that is, if it does not change the plane at all, then C is an inversion.

Exercises

1. If two polarities have in common a self-polar triangle ABC, where A is an actual point and B and C are two conjugate imaginary points on the ideal line defined by the elliptic involution set up on the ideal line by any circle in the plane, show that the quadric transformation arising from the two polarities is a circular transformation.

2. A plane is reflected across a fixed line lying in it; that is, each element is replaced by its symmetrical element with respect to this line. The plane is then inverted with respect to a fixed circle whose center lies on the fixed line. Show that the resultant transformation is a circular transformation.

3. Invert a hyperbola together with its asymptotes. Draw the inverse curve: (*a*) when the center of inversion is outside the hyperbola; (*b*) inside the hyperbola; (*c*) on the hyperbola; (*d*) at the center of the hyperbola.

168. General Note.—The most general quadric transformation is a birational point-transformation between two planes such that, in general, the lines in one plane correspond to conics in the other. There are three fundamental points in each plane. Lines passing

through a fundamental point in one plane are transformed into lines passing through a fundamental point in the other.

The fundamental points in either plane may be all distinct, two coincident, or all three coincident; they may be all real or two of them conjugate imaginary.

When the planes are superposed, the fundamental points in one plane may be all distinct from the fundamental points in the other; or the planes may have one, two, or three fundamental points in common.

In the last case we have the involutoric transformations with which we began in Art. 163.

The inversions were first considered by Dandelin (1822). If the points of a sphere are projected from the extremities of a diameter, upon a plane perpendicular to the diameter, corresponding points in the plane are inverse with respect to the circle which the plane cuts from the sphere. In this way, one or the other of the two inversions is obtained according as the plane cuts the sphere in a real circle or not.

Some traces of inversion are found earlier in the writings of Vieta (1600) and Fermat (1679).

The general theory of circular transformations is due to Möbius (1852–55), who called it **Kreis-Verwandschaft.**

This theory is of special importance in the theory of functions of the imaginary variable and its application to mathematical physics, since linear transformations of the imaginary variable are represented geometrically by circular transformations in a real plane.

The inversions were the first quadric transformations discovered. Those arising from two polarities were discovered by Magnus (1832–33). The perspective quadric transformations were discovered by Bellavitis (1838).

The more general quadric transformations have been studied by many geometricians and from various points of view, notably: Plücker (1830–35), Steiner (1832), and Seydewitz (1846).

The most general birational point-transformations between two planes were discovered by Cremona (1863–65) and are called after him, **Cremona transformations.** The quadric transformations are special cases of Cremona transformations.

INDEX

(Numbers refer to pages.)

A

Affinity, 170
  corresponding conics in, 172
  fundamental property of, 170
Algebraic equations of conics, 100
Anti-polarity, 190
Apollonius, 36
  circle of, 34
  theorem of, 174
Area, of parabolic segment, 173
  of ellipse, 175
Assumption, fundamental, 5
Asymptotes, 66
Asymptotic cone, 110
Axis, of a cone, 200
  of a conic, 97
  of a cylinder, 104
  of perspectivity, 22, 166

B

Bellavitis, 209
Brianchon, 80
  theorem of, 68
Bundle of lines, of planes, 2

C

Center, of conics, 94
  of perspectivity, 22, 166
  of projectivity, 118
  of surfaces, 110
Chains, of perspectivity, 37, 114, 160
Chord of contact, 83
Circle, of Apollonius, 34
  auxiliary, 156
  director, 158
Circular involution, 132
  transformations, 207
Circumscribing tetrahedrons, 109
Classification, of conics, 66
  of polarities, 187
  of projectivities, 120
  of surfaces, 109
Coaxial planes, 22
Collinear points, 22
Collineation, 161
  limiting lines of, 168
  self-corresponding elements, 179
Complete, n-edge, 13
  n-face, 13
  n-plane, 14
  n-point, 12, 14
  n-side, 13
Concurrent lines, 22
Configuration, of Desargues, 15
  of Pappus, 70
Confocal, elements, 196
  conics, 157
Cones, 79, 199
Conform transformations, 206
Congruence, 178
Conic sections, 66
Conjugate, of an ideal point, 30
  diameters, 94
  diametral planes, 104
  points and lines, 86

Construction of confocal elements, 196
of curves and envelopes, 62
of cyclic projectivities, 121
of foci, 146
of harmonic ranges, 27
of perspectivity, 166
of polarity, 184
of poles and polar lines, 84
Continuity, theorem of, 43
principle of, 77
Converse theorems, 22, 40, 69, 122
Coplanar figures, 22
Correlation of geometric figures, 16
Cremona, 209
Cross-ratio, 31
of harmonic range, 32
Curves, classification of, 66
of second order, 56
Cyclic planes, 201
projectivities, 121
Cylinders, 79

D

Dandelin, 209
Dedekind postulate, 42
Desargues, 15
configuration of, 15, 22
theorem of, 20, 134
Determination of projective relationship, 50, 163
Diagonal pyramid, 92
triangle, 91
Diameters, 94
Diametral planes, 104
Dimensions of primitive forms, 19
Directrices, 149
Duality, 161
in a plane, 8
in space, 9
principle of, 10

E

Elements, the, 1
common, of double polarity, 193
confocal, of double polarity, 196
double, of projectivity, 119
double conjugate, of double polarity, 194
dual, 8
fundamental, 202
homologous, 16
ideal, 5
reciprocal, 8
self-corresponding, 44, 116
separated, 39, 40
Elementary forms, 55
relations among, 64
totality of, 66
Envelope, 56
Ellipse, 66
Equation, of circle, 102
of ellipse, 100
of hyperbola, 100
of parabola, 103

F

Fermat, 209
Field of points, of lines, 2
Figures, complete, 12
coplanar, 22
geometric, 16
simple, 10
in space, 13
Focal radii, 150
axes, 201
Focus, 146
Forms, elementary, 55
primitive, 2

G

Generation of elementary forms, 55

Generation of elementary forms of particular conics and envelopes, 78
Generator of ruled surface, 105
Geometric figures, 16
mean, 33
Gergonne, 15

H

Harmonic elements, 112
conjugates, 27
mean, 32
pencils, 28
ranges, 25
scales, 38
Hyperbola, 66
Hyperboloids, 110

I

Imaginary elements, 141
lines, 138, 141
planes, 139
points, 137
Infinitely distant, 6
Invariant of perspectivity, 167
Inversion, 204
Involution, 127, 168
circular, 132
determined by quadrangle, 133
determined by quadrilateral, 133
elliptic, 128
hyperbolic, 128
on a sheaf of rays, 132
on a straight line, 130
parabolic, 129

K

Kreis-Verwandshaft, 209

L

Limiting lines of a collineation, 168
Linear complex, special, 3

M

Magnus, 209
Möbius, 54, 209

N

Normal conjugate rays, 30
Notation, 2, 19

O

Order, effect of, 26
Orthocenter, 137
Orthogonal circles, 34
Orthogonally correlated bundles, 160

P

Pappus, 36
configuration of, 70
Parabola, 66
Parallel, 5
Pascal, 80
theorem of, 68
Pencil, harmonic, 28
Pentagon theorem, 73
Perspective position, 16, 113, 159
Perspectivity, 165
harmonic, 168
invariant of, 167
Planes, cyclic, 201
diametral, 200
of symmetry, 104
principal, 104
Plücker, 209
Point of contact, 58

Point-row, 2
Polar figures, 89
  planes, 108
Polarity, absolute, 192
  double, 192
  in a bundle, 189
  in a plane, 183
  orthogonal, 190
  self-conjugate elements of, 185
Poles and polar lines, 81
  special positions of, 83
Pole-rays and polar planes, 91
Poncelet, 15
Postulate, Dedekind, 42
Primitive forms, of the first kind, 37
  of the second kind, 159
Principle of duality, 10
  of continuity, 77
Projection, central, 1
  and section, 3
Projective relationship, 38, 114, 160
  determination of, 50, 115, 163
Projectivity, 38, 165
  cyclic, 121
  elliptic, 120
  hyperbolic, 120
  parabolic, 120
Projector, 3
  of regulus, 65
Properties of curves and envelopes, 57
  of the similitude, 176
  reflection, 154

Q

Quadrangle, complete, 12
  theorem, 76
Quadrangles in perspective position, 23
Quadric transformations, 202
  perspective, 203
Quadrilateral, complete, 13

R

Radical axis, 177
Ranges, harmonic, 25
Reflection properties, 154
Regulus, 57

S

Section, 3
  of regulus, 65
  of surface, 106
Self-conjugate elements, 185, 190
Self-corresponding elements, 44, 179
Self-polar figures, 91
Seydewitz, 209
Sheaf, of lines or planes, 2
  of planes of second class, 56
Similarly projective, 79
Similitude, 176
  inverse points, 177
  properties of, 176
Simple n-edge, 13
  n-face, 13
  n-point, 11
  n-side, 11
Space, of points, of rays, of planes, 3
Steiner, 93, 209
Surfaces, ruled, 105
  classification of, 109
Superposition, 44, 116

T

Tangent cones, 108
  line, 58
  lines and planes, 107
Theorem, of Apollonius, 174
  of Brianchon, 68

Theorem, of continuity, 43
of Desargues, 20, 134
of Pascal, 68
of Von Staudt, 46
Theory of involution, 127
Trace, 3
Transformation, birational, 207
circular, 207
conform, 206
Transformation, Cremona, 209
quadric, 202

V

Vertices of a conic, 99
Vieta, 209
Von Staudt, 54

